Logic: another thing that penguins aren't very good at.

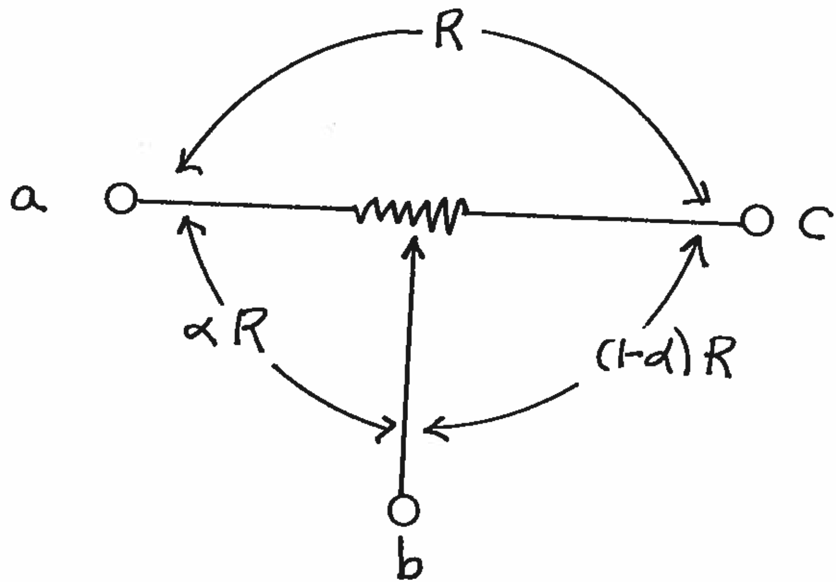


FIGURE 1.1 — A potentiometer is a three-terminal circuit element

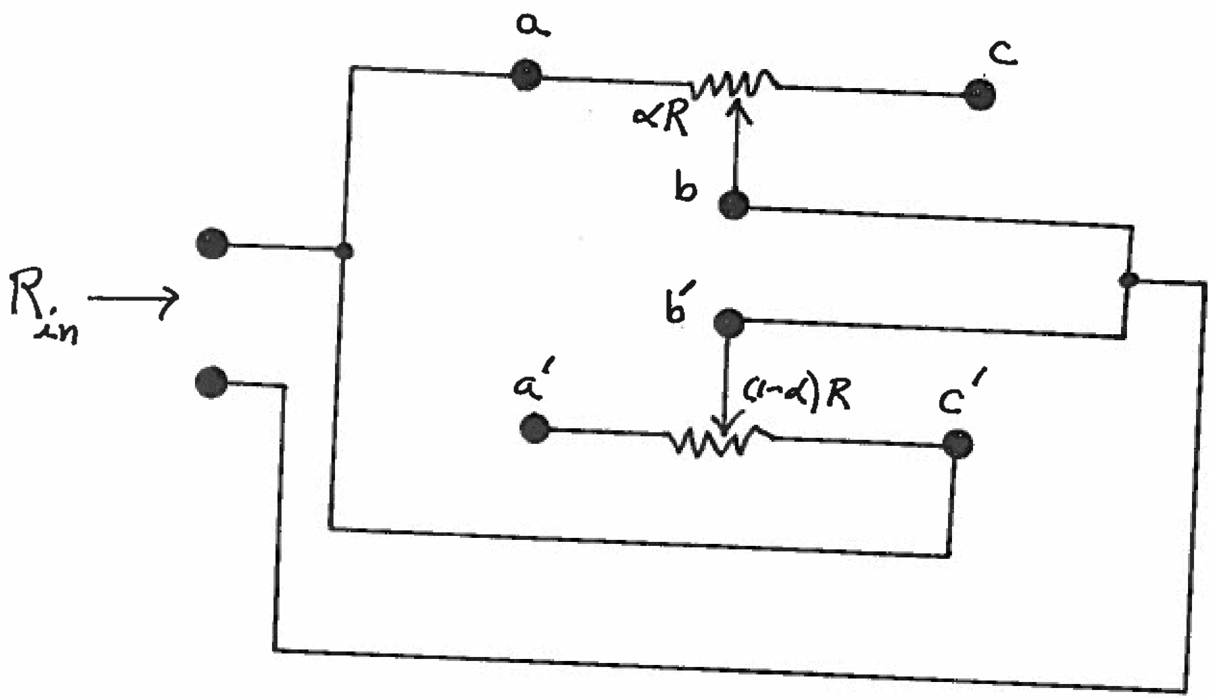
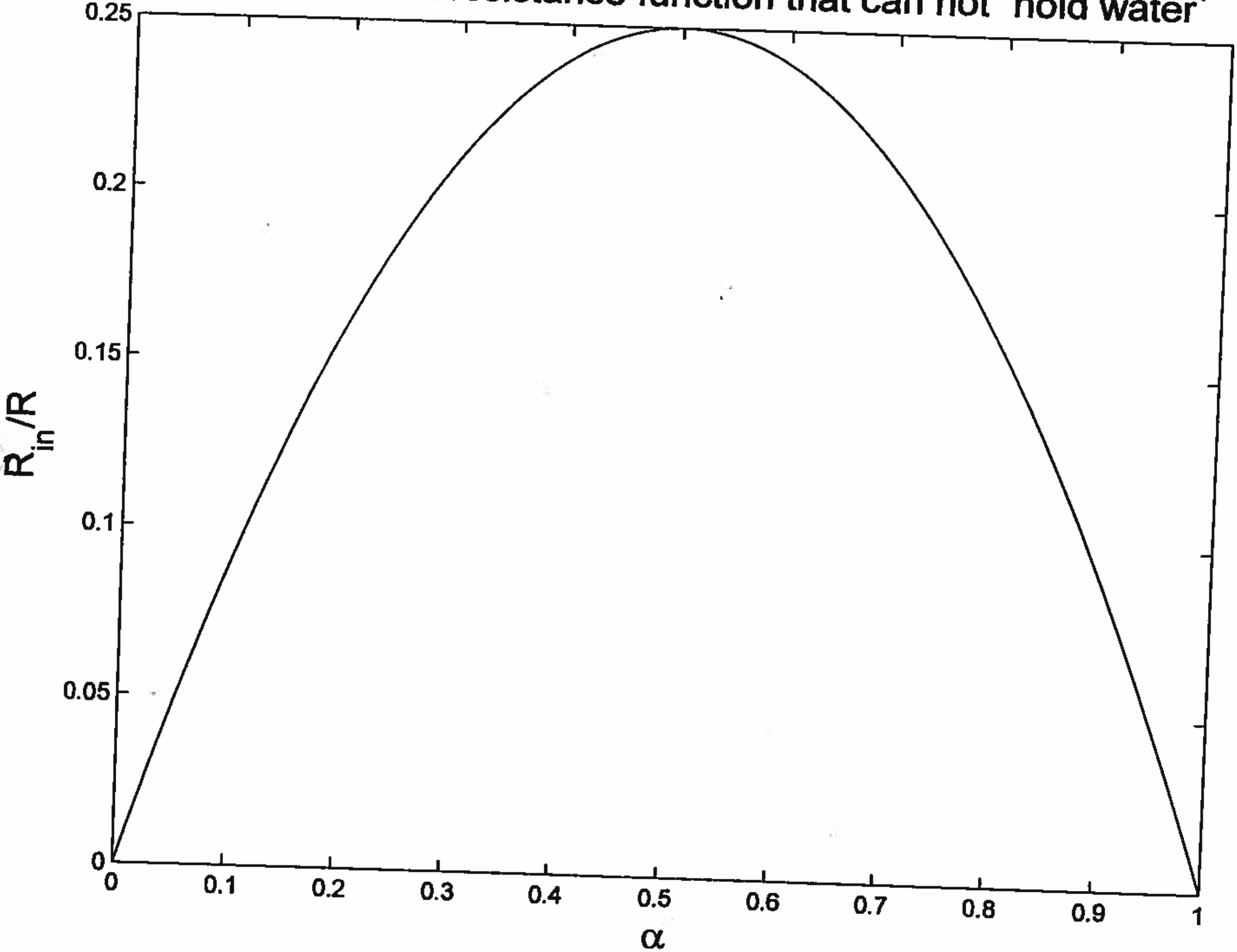


FIGURE 1.2 — Two potentiometers in parallel

FIGURE 1.3 - A resistance function that can not `hold water`



Puzzle 1

On the table before you are three small boxes, labeled **A**, **B**, and **C**. Inside each box is a colored plastic chip. One chip is red, one is white, and one is blue. You do not know which chip is in which box. Then, you are told that of the next three statements, *exactly one* is true:

(a) box **A** contains the red chip;

(b) box **B** does not contain the red chip;

(c) box **C** does not contain the blue chip.

You do not know which of the three statements is the true one. From all this, determine the color of the chip in each box. Puzzle 1 can be solved by some very careful reasoning that I'll show you at the end of this chapter (but don't peek until you've given it a good try yourself); it can also be solved through the *routine* application of Boolean algebra.

Now, just to really convince you that Boolean algebra is a most powerful tool, let me ask you to consider the next three puzzles, ones that I feel confident you will *not* be able to solve with simply "some very careful reasoning" or, at least, not until you've expended considerable mental effort. And yet, as we proceed through the book, I'll show you how they too will easily yield to routine Boolean algebraic analysis.[3]

Puzzle 2

The local truant officer has six boys under suspicion for stealing apples. He knows that only two are actually guilty (but not which two), and so he questions each boy individually.

(a) Harry said "Charlie and George did it."

(b) James said "Donald and Tom did it."

(c) Donald said "Tom and Charlie did it."

(d) George said "Harry and Charlie did it."

(e) Charlie said "Donald and James did it."

(f) Tom couldn't be found and didn't say anything.

(g) Of the five boys interrogated, four of them each correctly named *one* of the guilty.

(h) The remaining boy lied about both of the names he gave.

Who stole the apples?

Puzzle 3

Alice, Brenda, Cissie, and Doreen competed for a scholarship. "What luck have you had?" someone asked them.

Said Alice: "Cissie was top. Brenda was second."

Said Brenda: "No, Cissie was second, and Doreen was third."

Said Cissie: "Doreen was bottom. Alice was second."

Doreen said nothing.

Each of the three girls who replied made two assertions, of which only one was true. Who won the scholarship? More generally, in what position did each of the four girls finish?

Puzzle 4

Four hunters, A, B, C and D, occupied a camp for seven days. (1) On days when A hunted, B did not. (2) On days when B hunted, D also hunted, but C did not. (3) On days when D hunted, A or B hunted. No two days were identical in who hunted and who didn't. On how many days did D hunt, and with whom?

Okay, have you solved Puzzle 1? If not, here's how to do it 'with reasoning.' (By the end of Chapter 4 we'll have solved all four puzzles with the techniques of Boolean algebra.) Since we are told only one of the three statements is true, then we can attack the problem as follows: Take each one of the statements, in turn, as the true one, and *reverse* the other two. If we have selected the correct true statement, then we'll have three true statements. Since there are only three statements in all, we only have to do this three times. For each group of 'corrected' three statements we can then see if what they say, collectively, makes sense. So,

Case 1: Take (a) as true, and (b) and (c) as false. Then, with reversals, we have

(a1) box **A** contains the red chip;

(b1) box **B** contains the red chip;

(c1) box **C** contains the blue chip.

This is, of course, obvious nonsense as (a1) and (b1) cannot both be true.

Case 2: Take (b) as true, and (a) and (c) as false. Then, with reversals, we have

(a2) box **A** does not contain the red chip;

(b2) box **B** does not contain the red chip;

(c2) box **C** contains the blue chip.

Since box **C** has the blue chip, then the red and white chips are in boxes **A** and **B**. In particular, one of those two boxes *must* have the red chip, but (a2) and (b2) deny that. Thus, Case 2 is also nonsense.

Case 3: Take (c) as true, and (a) and (b) as false. Then, with reversals, we have

(a3) box **A** does not contain the red chip;

(b3) box **B** contains the red chip;

(c3) box **C** does not contain the blue chip.

This works. (b3) says **B** has the red chip. That leaves the blue and white chips for **A** and **C**. (c3) says **C** does not have the blue chip, so **C** must have the white chip. Thus, **A** must have the blue chip, which is consistent with (a3).

The author of a well-known science fiction story,[4] in which the narrator is a college math major, opens his tale with the student complaining about his courses. In particular, his class in logic generates the lament "If it seems to make sense it isn't mathematical logic!" By the time you finish this book I hope you'll reject that sentiment and, instead, agree with me that if mathematical logic is about anything, it *is* about 'making sense.'

Okay, have you solved the 'two mathematicians' puzzle? If not, take a look at the final note.[5]

ages			sum of ages
2	6	3	11
4	3	3	10
2	2	9	13
4	1	9	14
3	1	12	16
2	1	18	21
1	6	6	13
1	1	36	38

FIGURE 3.1.1 - Boole was in London June and July of 1864, just months before his death. While there, he stopped-in at the famous London School of Photography at 174 Regent Street, one of the pioneers in commercial Victorian photography, and had this full-length portrait taken.

Photo reproduced by arrangement with the Boole Library,
Special Collections and Archives, University College,
Cork, Ireland

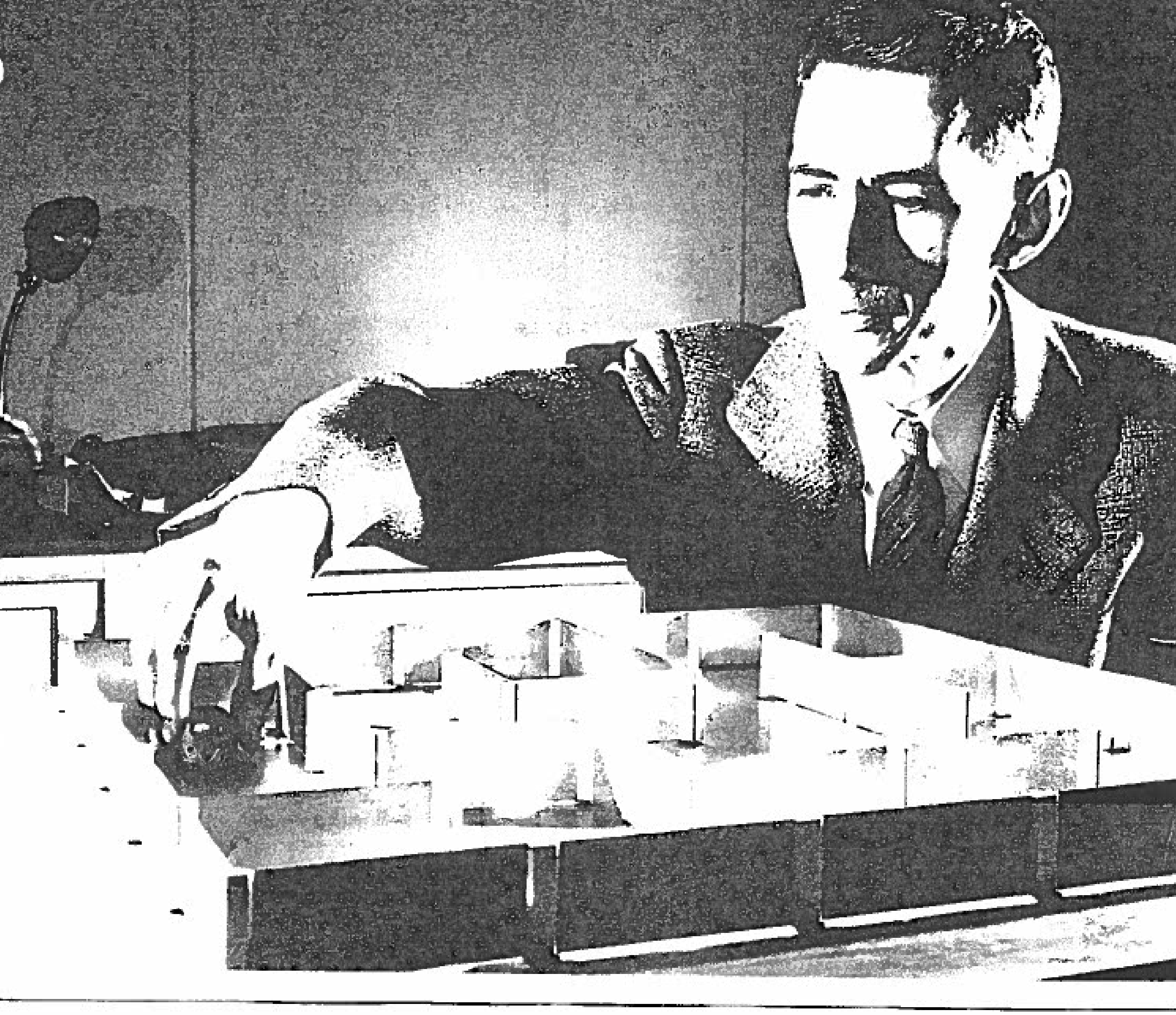


FIGURE 3.2.1 – This photograph, taken in 1952, shows Shannon with Theseus, his maze-solving 'mouse' built in 1950. The mouse was named in honor of the character from Greek mythology who, after killing the Minotaur in the monster's maze (the Labyrinth), found his way back out because he had unrolled a ball of string behind him on the way in. The mouse was moved through a 5-by-5 square, reconfigurable maze by an electromagnet mounted on wheels positioned beneath the floor of the maze. Electric motors powered the wheels, and the motors in turn were controlled by a relay logic circuit (also beneath the floor). The mouse could 'explore' the maze according to a fixed strategy that Shannon built into the relay logic, 'learning' where the maze walls were by bumping into them. Eventually, the mouse (that is, the relay logic) learned to run, without bumping any wall, through the entire maze.

Photo reproduced by arrangement with the MIT Museum, Cambridge, MA

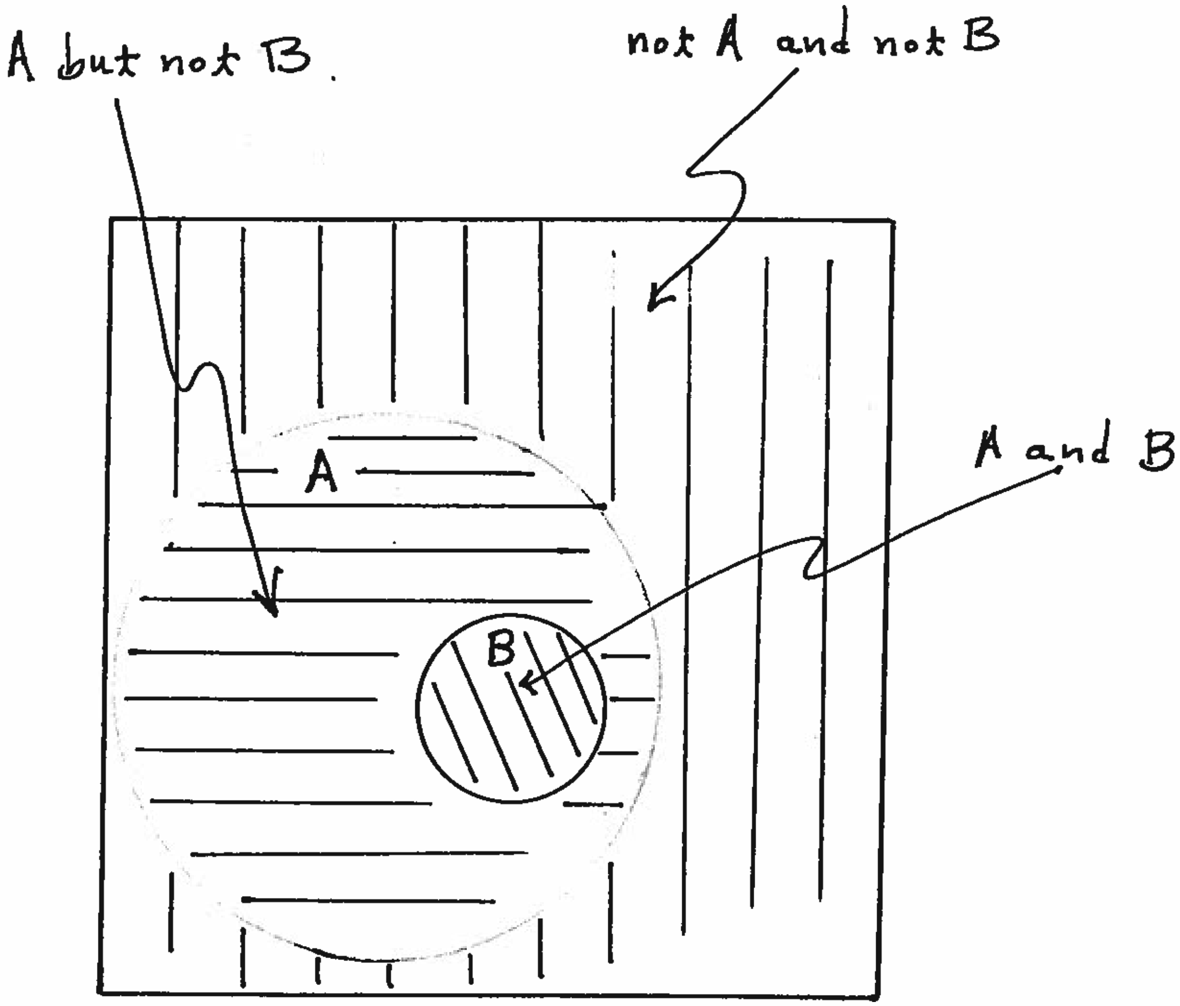


FIGURE 4.2.1 — Every element of **B** is an element of **A**

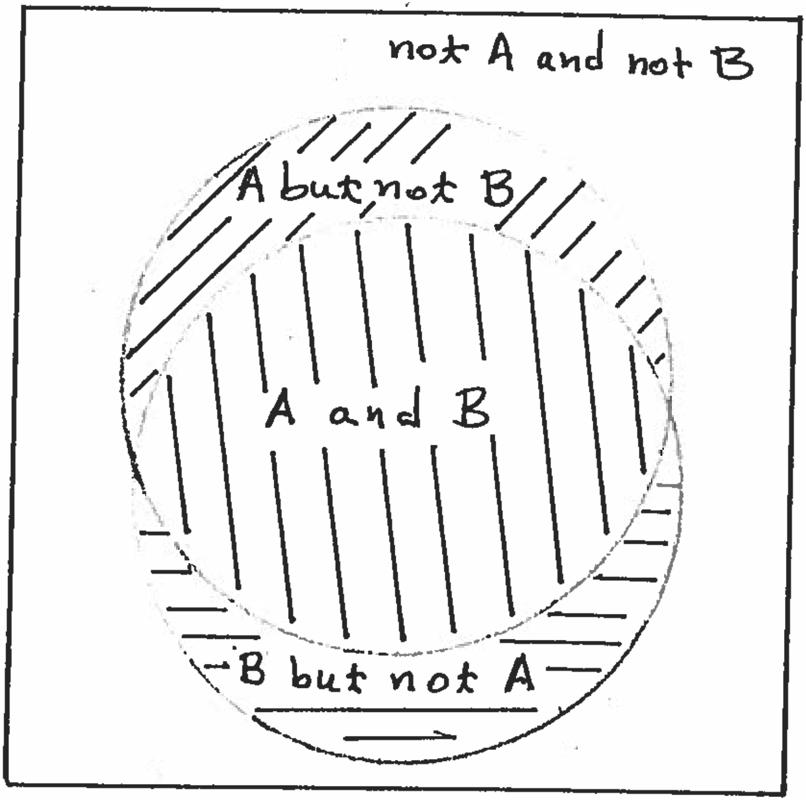


FIGURE 4.2.2 — The four general regions of two sets

$$(4.3.1)\quad xy = x \cap y.$$

$$(4.3.2)\ x + y = x \cup y.$$

$$
\begin{aligned}
(4.3.3) \qquad 0 \cdot 0 &= 0 \\
0 \cdot 1 &= 0 \\
1 \cdot 0 &= 0 \\
1 \cdot 1 &= 1
\end{aligned}
$$

$$
\begin{aligned}
(4.3.4) \qquad 0+0&=0\\
0+1&=1\\
1+0&=1\\
1+1&=1.
\end{aligned}
$$

$$
\begin{aligned}
(4.3.5) \qquad 0 \oplus 0 &= 0 \\
0 \oplus 1 &= 1 \\
1 \oplus 0 &= 1 \\
1 \oplus 1 &= 0.
\end{aligned}
$$

(4.4.1) $\quad A\bar{A} = 0,$

(4.4.2) $\quad AA = A,$

(4.4.3) $\quad A + \bar{A} = 1,$

(4.4.4) $\quad A + 1 = 1,$

(4.4.5) $\quad A + 0 = A,$

(4.4.6) $\quad A + A = A,$

(4.4.7) $\quad A + AB = A,$

(4.4.8) $\quad \overline{AB} = \bar{A} + \bar{B},$

(4.4.9) $\quad \overline{A + B} = \bar{A}\bar{B}.$

A	B	A + AB
0	0	0
0	1	0
1	0	1
1	1	1

A	B	$\overline{AB}$	$\bar{A} + \bar{B}$	$\overline{A + B}$	$\bar{A}\bar{B}$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	1	1	0	0
1	1	0	0	0	0

number of variables	number of functions
1	4
2	16
3	256
4	65,536
5	4,294,967,296
6	$1.84 \cdot 10^{19}$

$$A_r + B_r + C_r = 1, \tag{4.5.1}$$

$$A_r B_r C_b + \bar{A}_r \bar{B}_r C_b + \bar{A}_r B_r \bar{C}_b = 1 \tag{4.5.2}$$

$$\tilde{A}_r \breve{B}_r C_b + \tilde{A}_r B_r \bar{C}_b = 1. \tag{4.5.3}$$

$$A_r\bar{A}_r\bar{B}_rC_b + A_r\bar{A}_rB_r\bar{C}_b + B_r\bar{A}_r\bar{B}_rC_b + B_r\bar{A}_rB_r\bar{C}_b + C_r\bar{A}_r\bar{B}_rC_b + C_r\bar{A}_rB_r\bar{C}_b = 1. \tag{4.5.4}$$

$$\bar{A}_r B_r \bar{C}_b = 1 \tag{4.5.5}$$

$$(C + G)(D + T)(T + C)(H + C)(D + J) = 0 \tag{4.5.6}$$

$$(CD + GD + CT + GT)(TH + CH + TC + C)(D + J) = 0. \tag{4.5.7}$$

$$(4.5.8) \qquad CD = 0.$$

$$(4.5.9) \quad (C+G)(D+T)(T+C)(H+C)+(C+G)(D+T)(T+C)(D+J) + (C+G)(D+T)(H+C)(D+J)+(C+G)(T+C)(H+C)(D+J) + (D+T)(T+C)(H+C)(D+J)=1.$$

$$C_1B_2 = 0 \tag{4.5.10}$$

$$(4.5.11) \qquad C_1 + B_2 = 1.$$

$$\bar{B}_2 C_1 + B_2 \bar{C}_1 = 1. \tag{4.5.12}$$

$$\bar{C}_2 D_3 + C_2 \bar{D}_3 = 1. \tag{4.5.13}$$

From which it follows that

$$(4.5.14) \qquad \bar{A}_2 D_4 + A_2 \bar{D}_4 = 1.$$

we have the reduction to

$$(\bar{B}_2C_1\bar{C}_2D_3 + B_2\bar{C}_1\bar{C}_2D_3)(\bar{A}_2D_4 + A_2\bar{D}_4) = 1. \tag{4.5.15}$$

$$A_2\bar{D}_4\bar{B}_2C_1\bar{C}_2D_3 = 1. \tag{4.5.16}$$

row	A	B	C	D
1	0	0	0	0
2	0	0	0	1
3	0	0	1	0
4	0	0	1	1
5	0	1	0	0
6	0	1	0	1
7	0	1	1	0
8	0	1	1	1
9	1	0	0	0
10	1	0	0	1
11	1	0	1	0
12	1	0	1	1
13	1	1	0	0
14	1	1	0	1
15	1	1	1	0
16	1	1	1	1

A	B	C	D
–	–	–	–
0	0	0	0
0	0	1	0
0	1	0	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1

A	B	F
0	0	1
0	1	0
1	0	1
1	1	1

$$F = \bar{A}\bar{B} + A\bar{B} + AB. \tag{4.6.1}$$

$$F = (\bar{A} + A)\bar{B} + AB = \bar{B} + AB \tag{4.6.2}$$

$$(4.6.3) \qquad F = \bar{A}\bar{B} + A(\bar{B} + B) = \bar{A}\bar{B} + A.$$

$$F = A + \bar{B}. \tag{4.6.4}$$

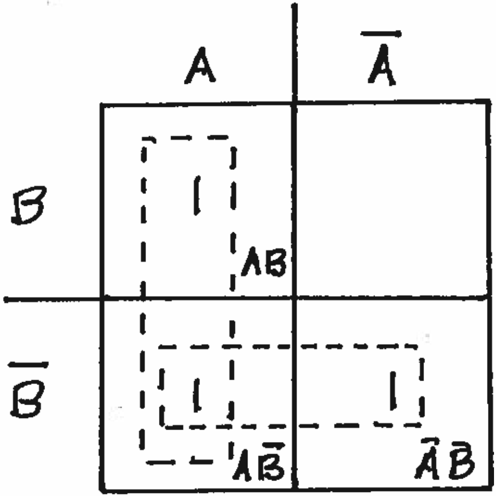


FIGURE 4.6.1 — The Karnaugh map for F

A	B	C	G
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

$$G = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + A\bar{B}C \tag{4.6.5}$$

$$G = \bar{A}\bar{B} + \bar{B}C. \tag{4.6.6}$$

$$\bar{G} = A\bar{C} + B \tag{4.6.7}$$

$$G = (\bar{A} + C)\bar{B}. \tag{4.6.8}$$

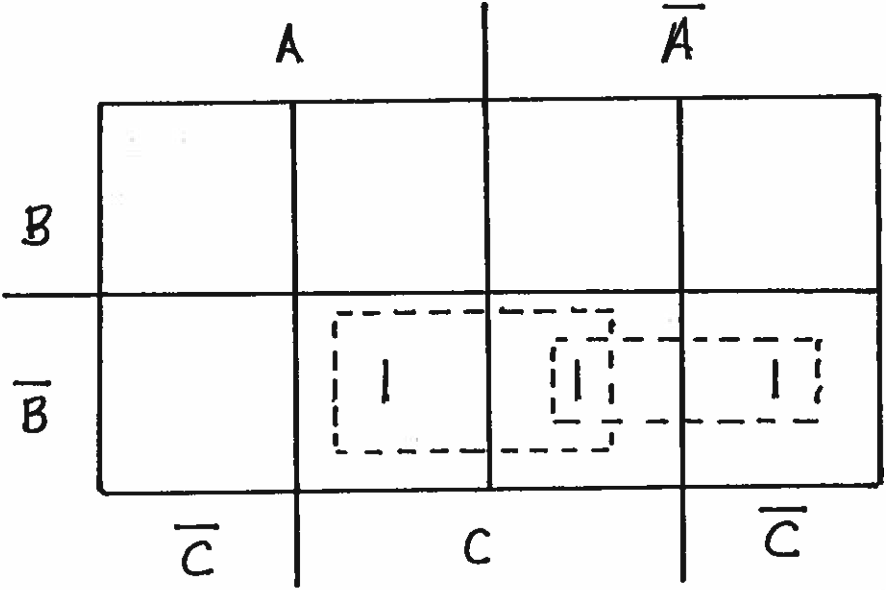


FIGURE 4.6.2 — The Karnaugh map for G

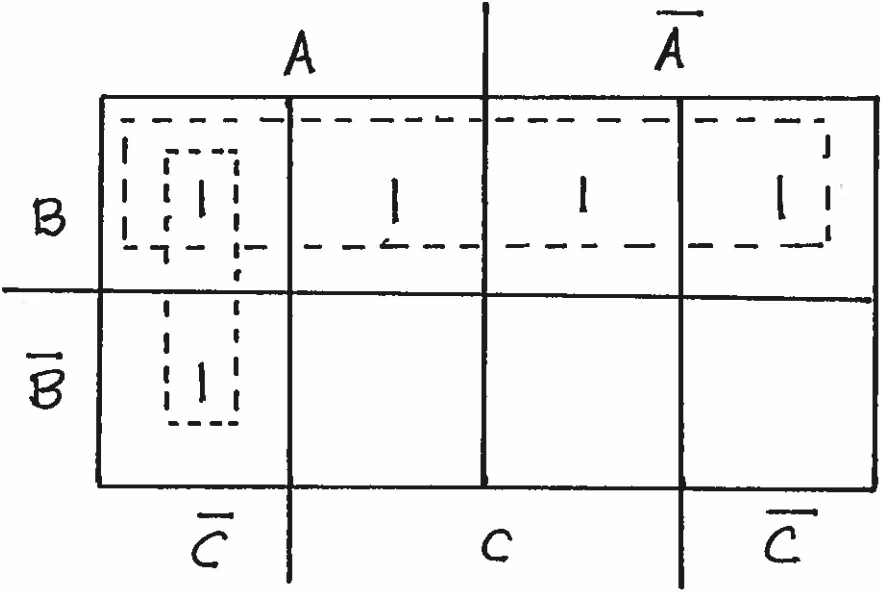


FIGURE 4.6.3 — The Karnaugh map for $\bar{G}$

A	B	C	D	H
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0,1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0,1
0	1	1	1	0
1	0	0	0	0,1
1	0	0	1	1
1	0	1	0	0,1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

$$H = \bar{A}B\bar{C}D + A\bar{B}\bar{C}D + AB\bar{C}\bar{D} + AB\bar{C}D + ABCD + (\bar{A}\bar{B}C\bar{D} + \bar{A}BCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}C\bar{D}). \tag{4.6.9}$$

$$(4.6.10)\ H = A\bar{C} + BD$$

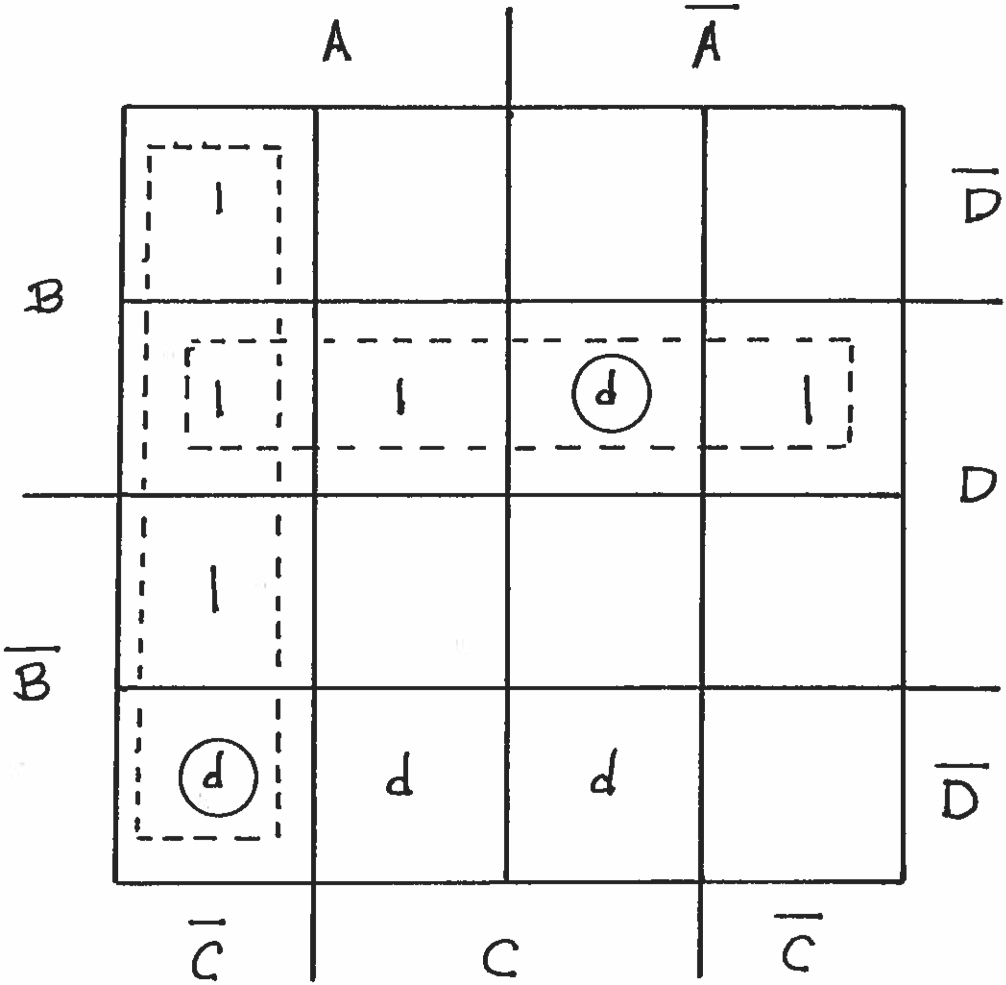


FIGURE 4.6.4 — The Karnaugh map for H

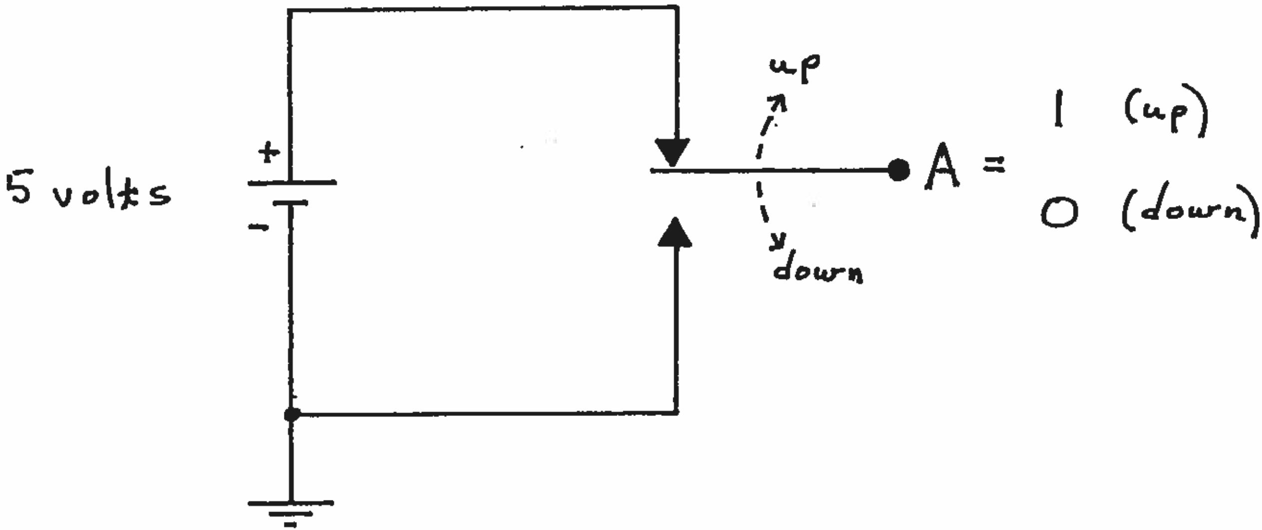


FIGURE 5.2.1 — A hand-operated switch

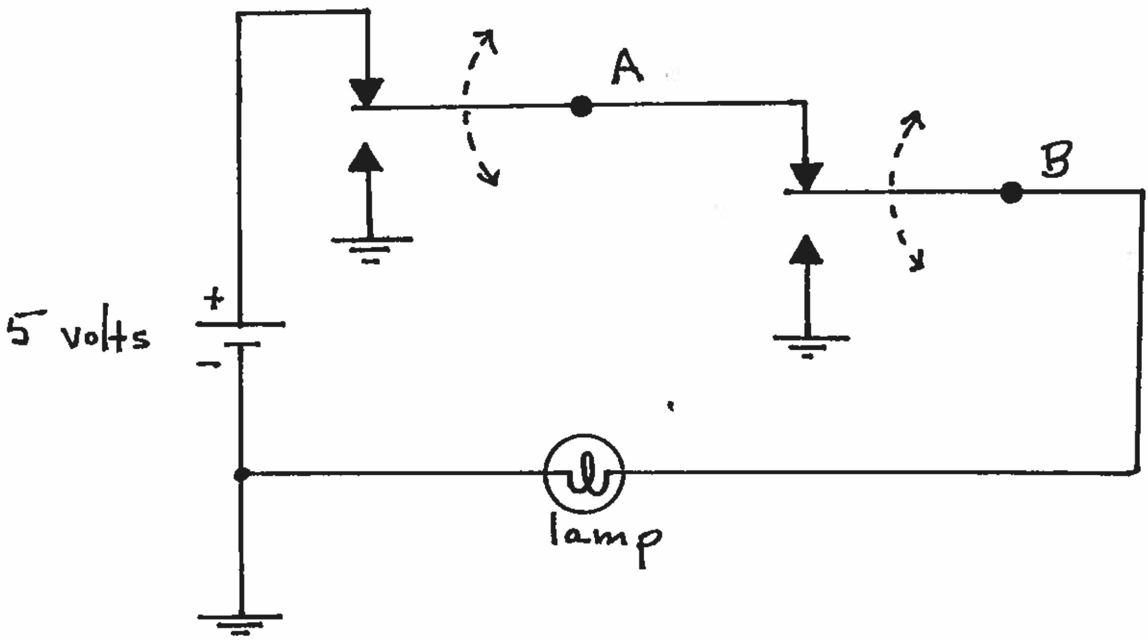


FIGURE 5.2.2 — *Series* means *and*

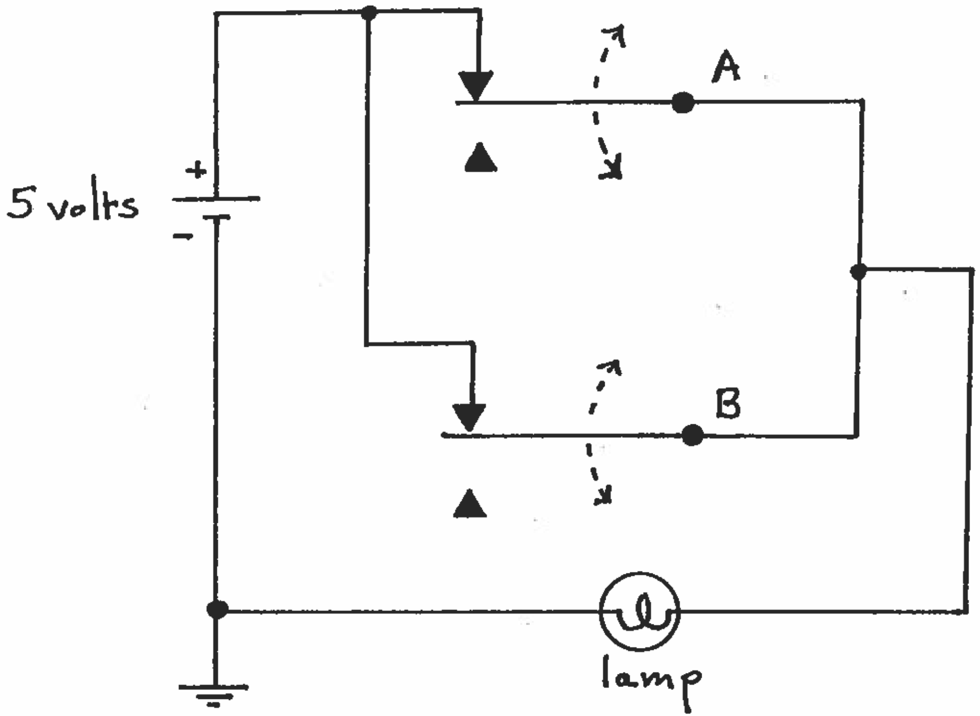


FIGURE 5.2.3 — *Parallel* means *or*

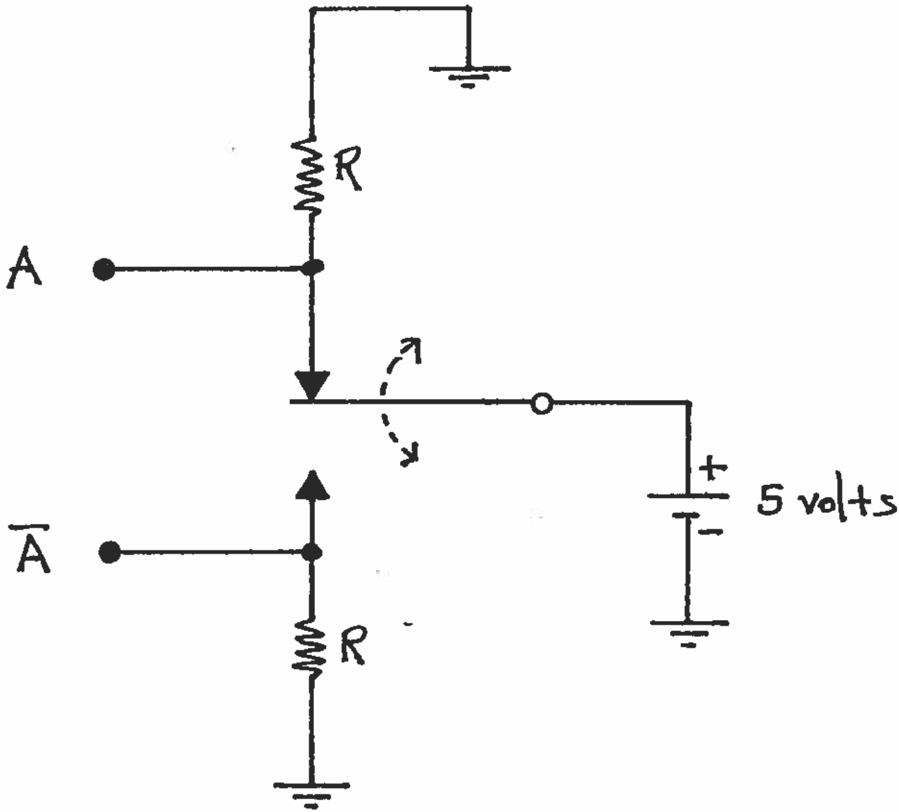


FIGURE 5.2.4 — One way to generate A and Ā

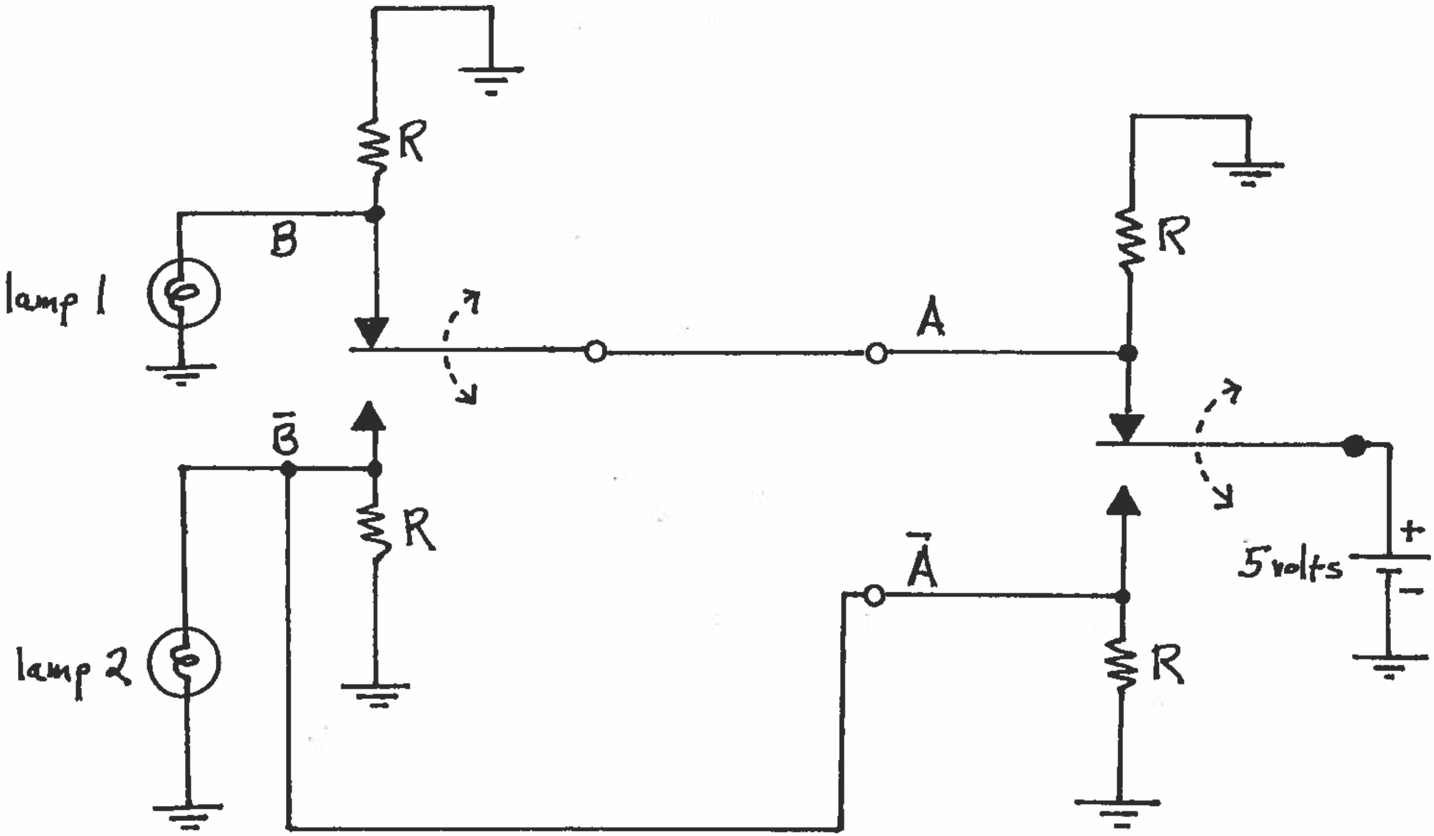


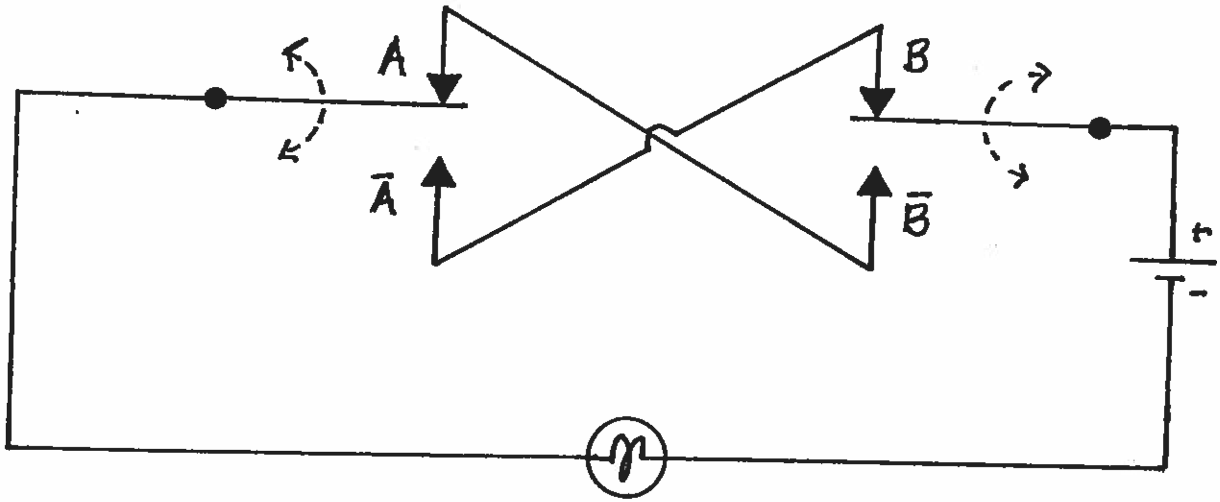
FIGURE 5.2.5 — A two-lamp circuit

A	B	L
0	0	0
0	1	1
1	0	1
1	1	0

$$L = \bar{A}B + A\bar{B}. \tag{5.3.1}$$

A	B	L
0	0	1
0	1	0
1	0	0
1	1	1

$$(5.3.2) \qquad L = \bar{A}\,\bar{B} + AB.$$



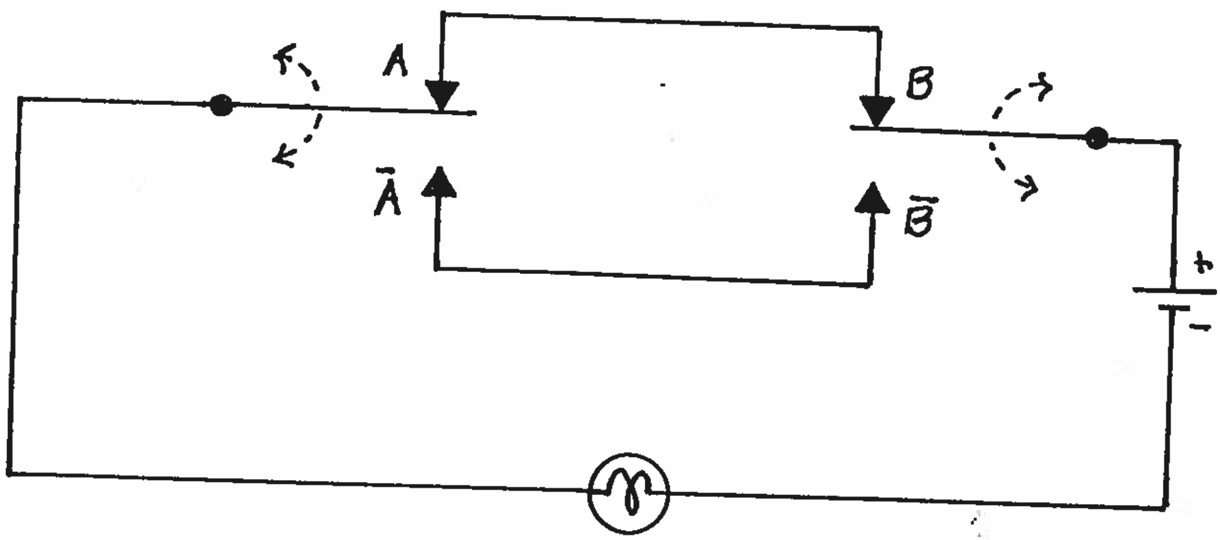


FIGURE 5.3.1 — Two solutions to the staircase light problem

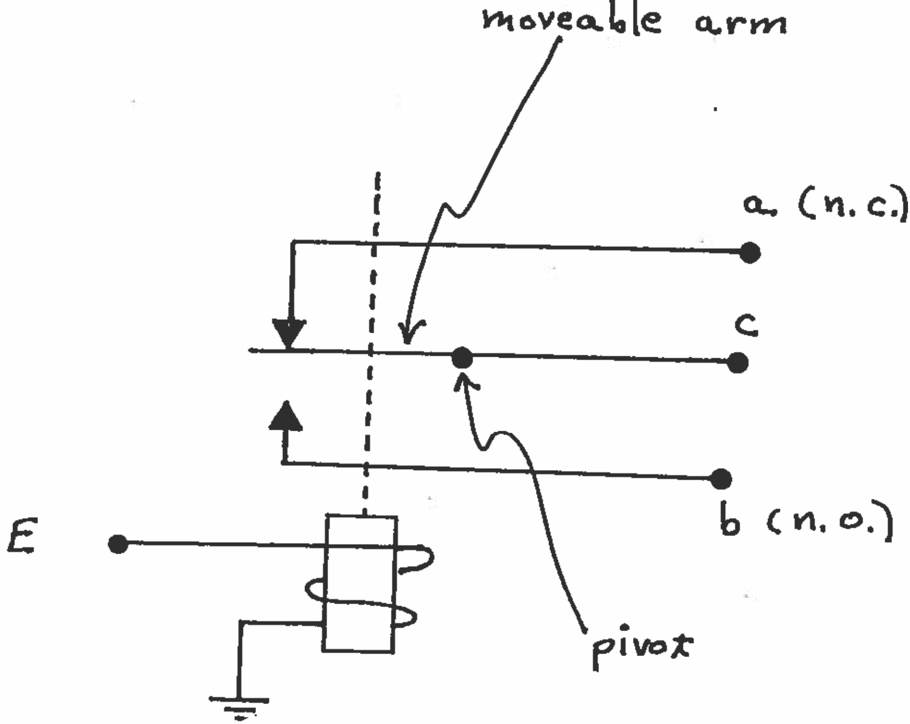


FIGURE 5.4.1 — An electromagnetic relay

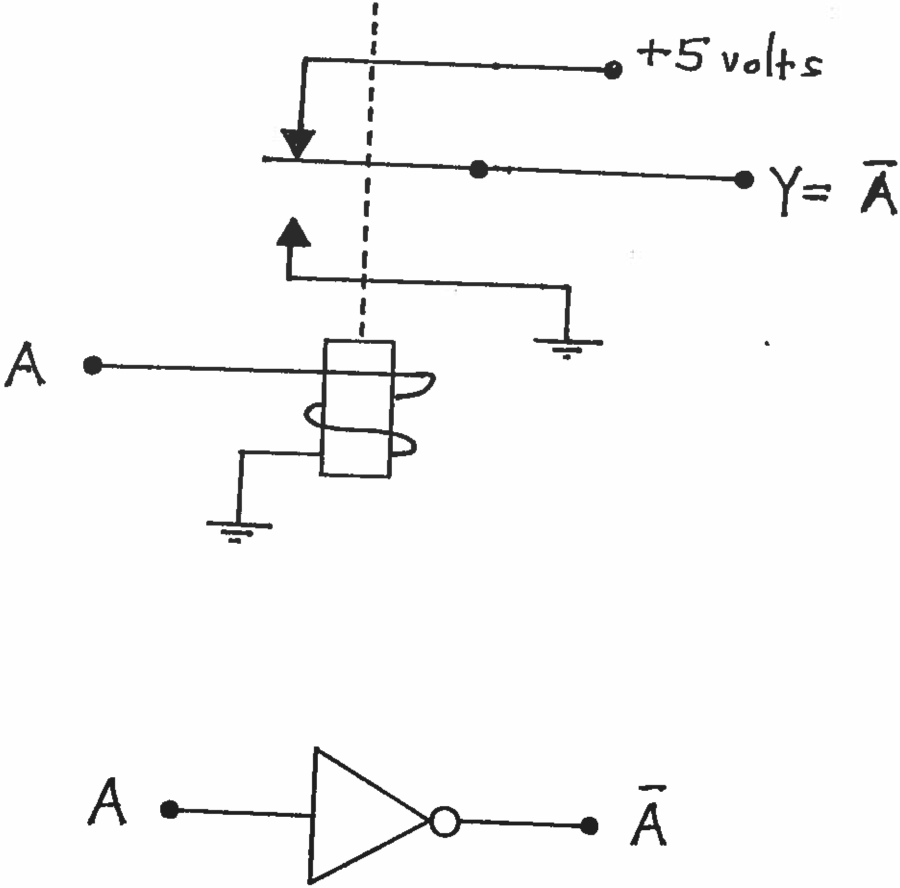


FIGURE 5.4.2 — The relay logical inverter (NOT gate)

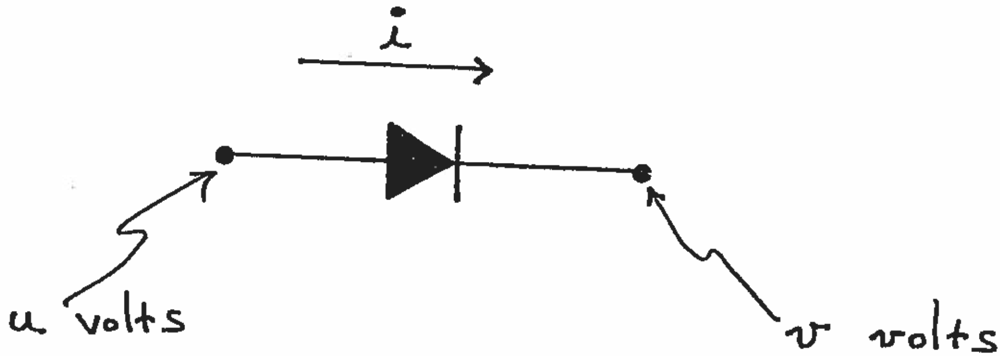


FIGURE 5.5.1 — The diode

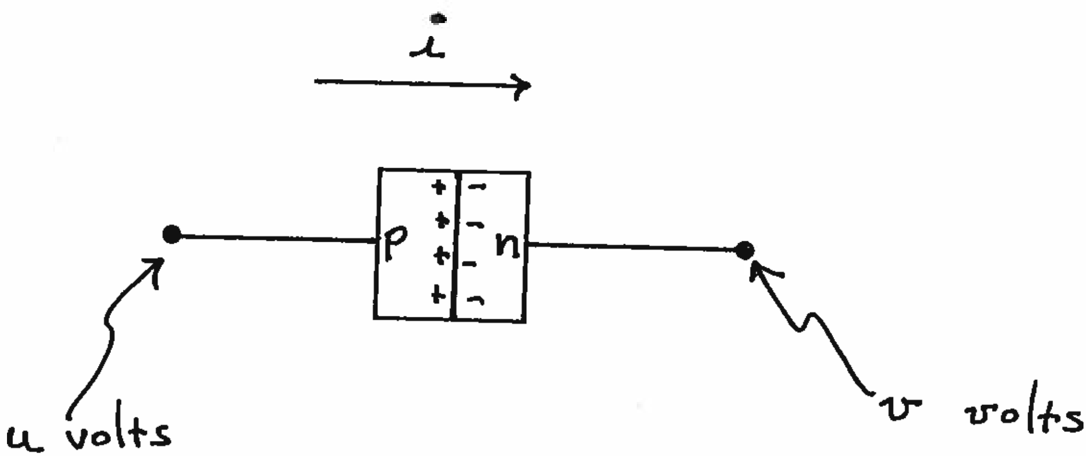


FIGURE 5.5.2 — The pn-junction

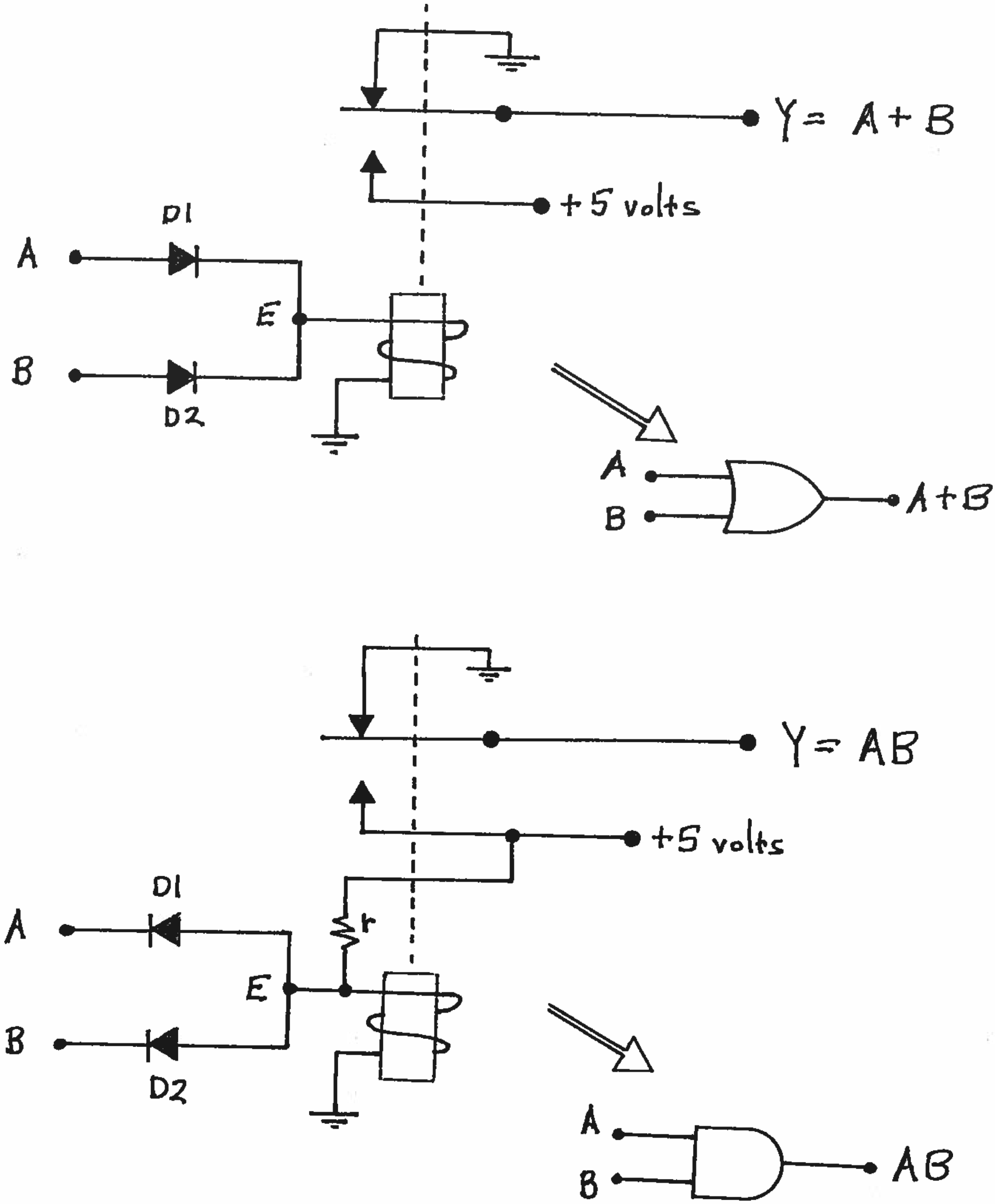


FIGURE 5.5.3 — The relay inclusive-OR and AND logic gates

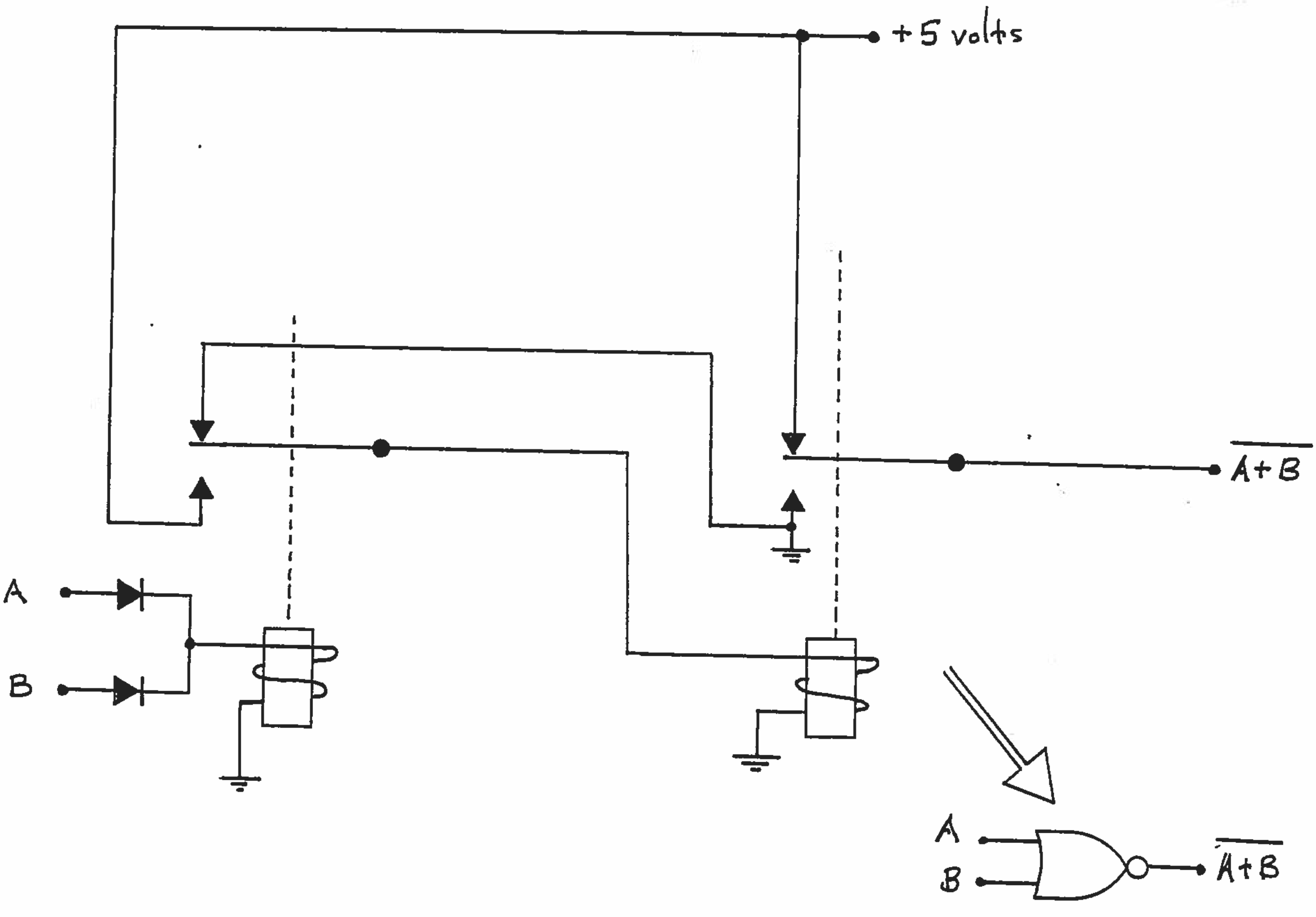


FIGURE 5.5.4 — The relay NOR logic gate

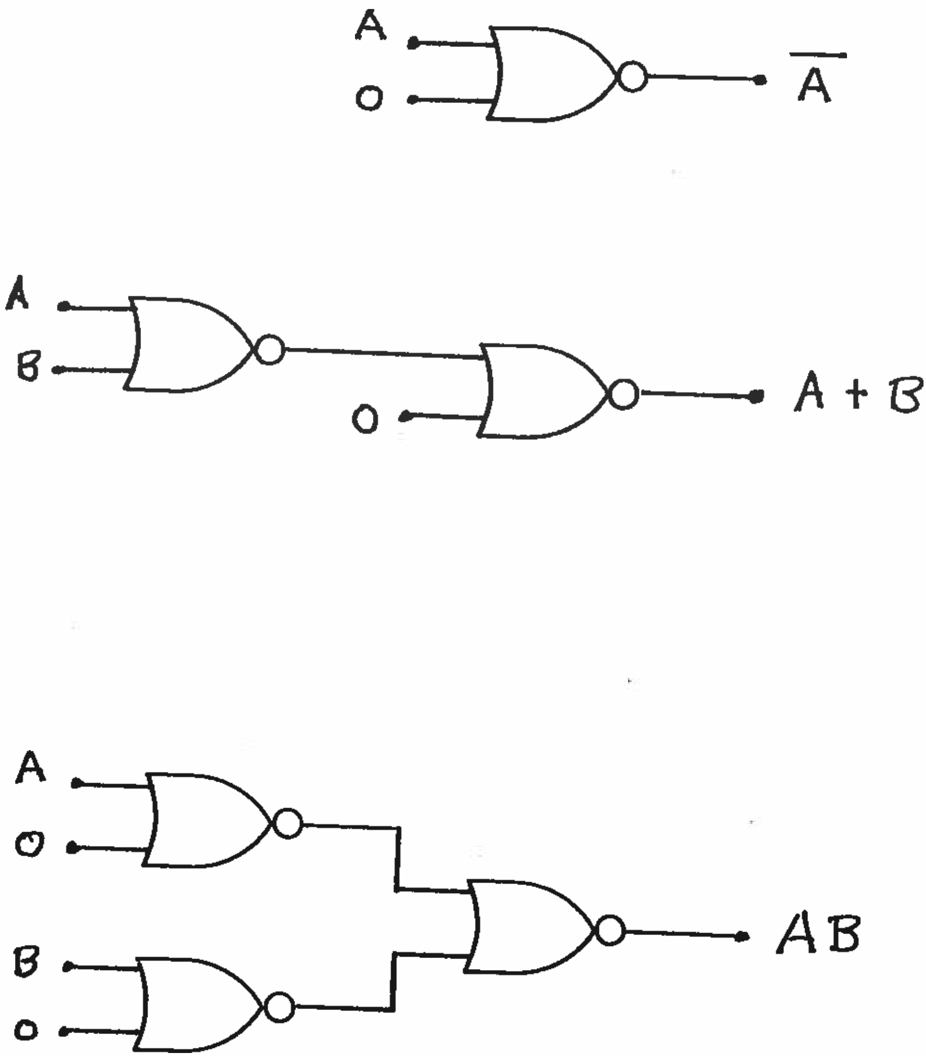


FIGURE 5.5.5 — NOT, Inclusive-OR, and AND logic gates from just NOR gates

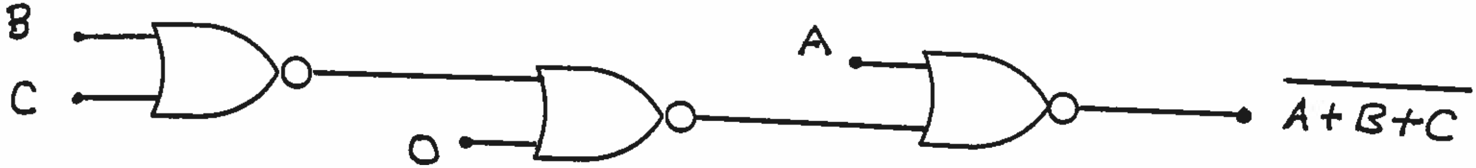


FIGURE 5.5.6 — Building a 3-input NOR from 2-input NORs.

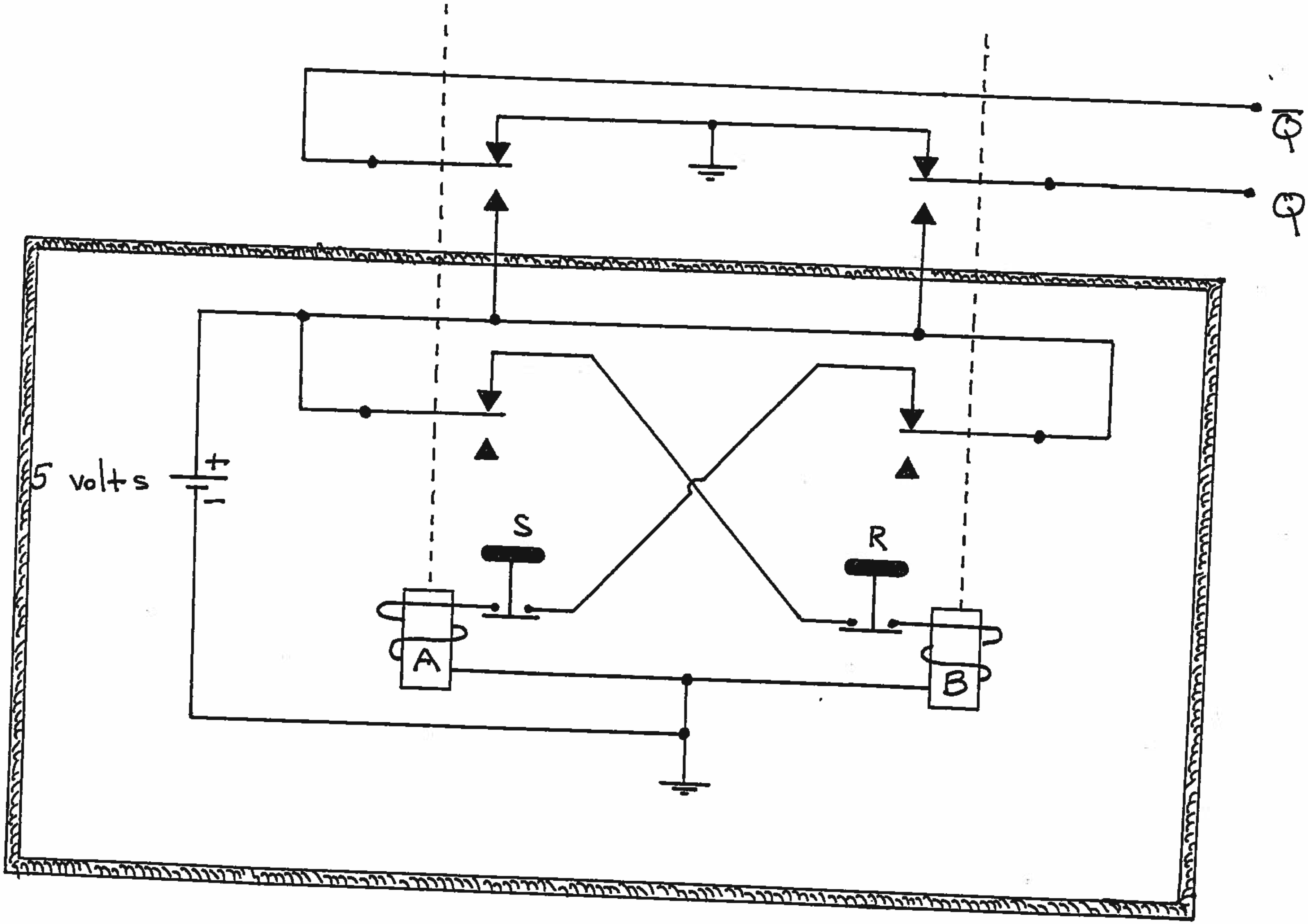


FIGURE 5.6.1 — The bi-stable relay latch

first flip	second flip
H	H
H	T
T	H
T	T

$$\sum_{i=1}^{4} P(s_i) = 1. \tag{6.2.1}$$

$$A = \{\text{"at least one head (H) occurs"}\}$$

and

$$B = \{\text{"both heads occur"}\}.$$

Then,

$$(6.2.2) \qquad P(A) = P(s_1) + P(s_2) + P(s_3) = \frac{3}{4}$$

while

$$(6.2.3) \qquad P(B) = P(s_1) = \frac{1}{4}.$$

Notice that

$$(6.2.4) \qquad P(AB) = \frac{1}{4}$$

(a) $P(AB) = \frac{n_{ab}}{N}$;

(b) $P(A) = \frac{n_a}{N}$;

(c) $P(B) = \frac{n_b}{N}$;

(d) $P(A|B) = \frac{n_{ab}}{n_b}$;

(e) $P(B|A) = \frac{n_{ab}}{n_a}$.

$$P(A + B) = P(A) + P(B) - P(AB). \tag{6.2.5}$$

$$P(B|A) = \frac{n_{ab}}{n_a} = \frac{n_{ab}/N}{n_a/N} = \frac{P(AB)}{P(A)} \tag{6.2.6}$$

$$(6.2.7) \qquad P(A|B) = \frac{n_{ab}}{n_b} = \frac{n_{ab}/N}{n_b/N} = \frac{P(AB)}{P(B)}.$$

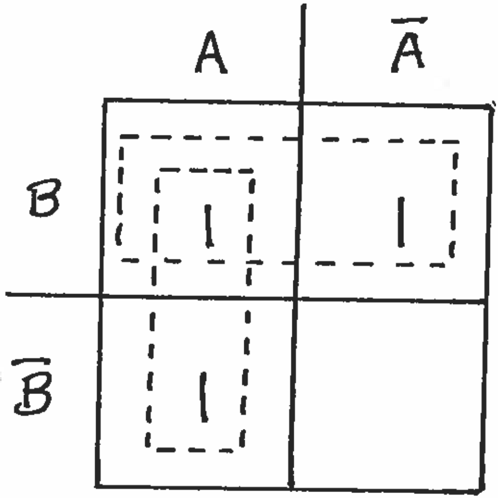


FIGURE 6.2.1 — The map of A + B covers AB *twice*

$$P(A) = P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B}). \tag{6.2.8}$$

$$P(B) = P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A}). \tag{6.2.9}$$

$$P(\bar{Y} \mid X) = \frac{P(X \mid \bar{Y})P(\bar{Y})}{P(X)} = \frac{P(X \mid \bar{Y})[1 - P(Y)]}{P(X)}. \tag{6.3.1}$$

$$\text{(6.3.2)} \qquad P(\bar{Y}|X) = \frac{P(X|\bar{Y})\left[1 - P(Y)\right]}{(1 - p)P(Y) + P(X|\bar{Y})\left[1 - P(Y)\right]}$$

(6.3.3) $$P(E_1 + E_2 + \cdots + E_k) \leq P(E_1) + P(E_2) + \cdots + P(E_k).$$

$$(6.4.1) \qquad P(S1) = 1 - (1 - p)^2.$$

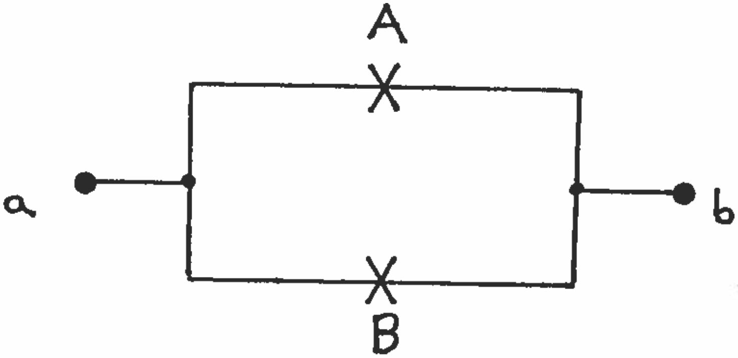


FIGURE 6.4.1 — Two crummy relays with make contacts in parallel

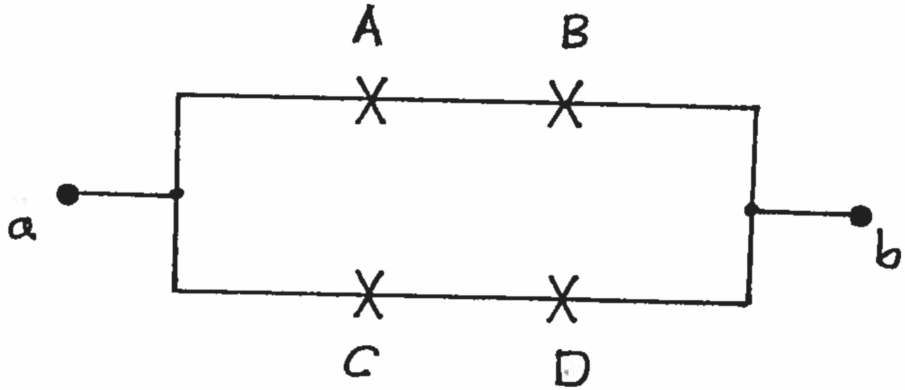


FIGURE 6.4.2 — Four crummy relays in series/parallel

$$P(\bar{Y}|X) = \frac{P(X|\bar{Y})[1 - P(Y)]}{(1 - p)P(Y) + P(X|\bar{Y})[1 - P(Y)]} \tag{6.3.2}$$

$$(6.3.3) \qquad P(E_1 + E_2 + \cdots + E_k) \le P(E_1) + P(E_2) + \cdots + P(E_k).$$

$$P(S3) = P(S3 \mid E = 1)P(E = 1) + P(S3 \mid E = 0)P(E = 0). \tag{6.4.3}$$

$$
(6.4.4) \quad \begin{aligned}
&\text{(a) } P(E = 1) = p \\
&\text{(b) } P(E = 0) = 1 - p \\
&\text{(c) } P(S3 \mid E = 0) = 2p^2 - p^4.
\end{aligned}
$$

$$(6.4.5) \qquad P(S3 \mid E = 1) = P(AB + CD + AD + CB).$$

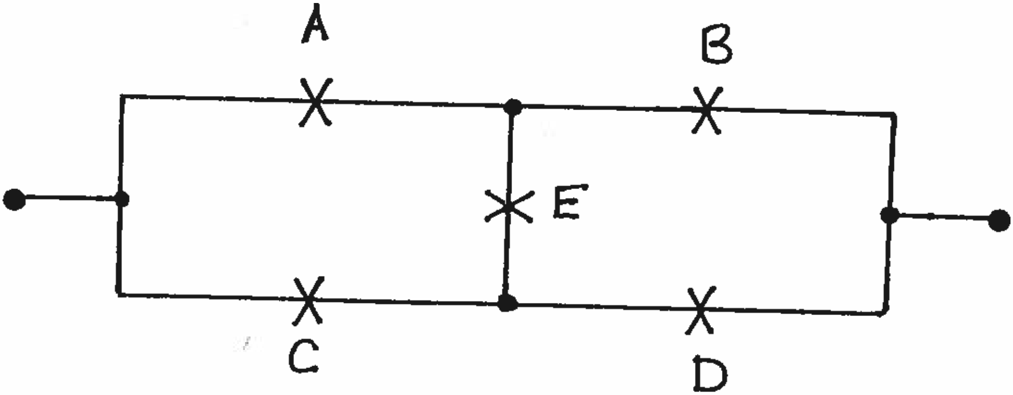


FIGURE 6.4.3 — A bridging arrangement

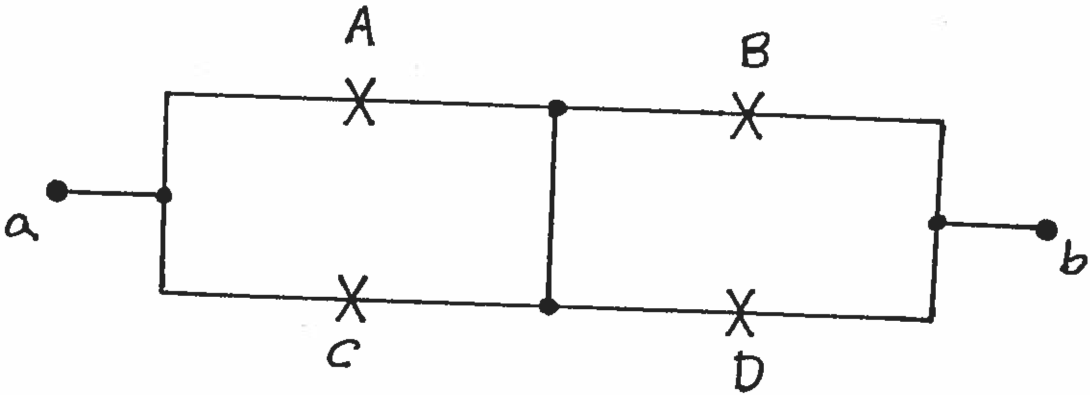


FIGURE 6.4.4 — The bridge arrangement when E = 1

as 1 *minus* the probability of the *inverse* ma

(6.4.6) $P(S3|E = 1) = 1 - P(\bar{A}\bar{C}) - P(\bar{B}\bar{D}) + P(\bar{A}\bar{B}\bar{C}\bar{D})$.

$$P(S3 \mid E = 1) = 4p^2 - 4p^3 + p^4. \tag{6.4.7}$$

$$P(S3) = 2p^2 + 2p^3 - 5p^4 + 2p^5, \tag{6.4.8}$$

	A	A	$\bar{A}$	$\bar{A}$	
	$\bar{C}$	C	C	$\bar{C}$	
B	1	1	1		$\bar{D}$
B	1	1	1		D
$\bar{B}$	1	1	1		D
$\bar{B}$					$\bar{D}$

FIGURE 6.4.5 — The Karnaugh map for P(S3 | E = 1)

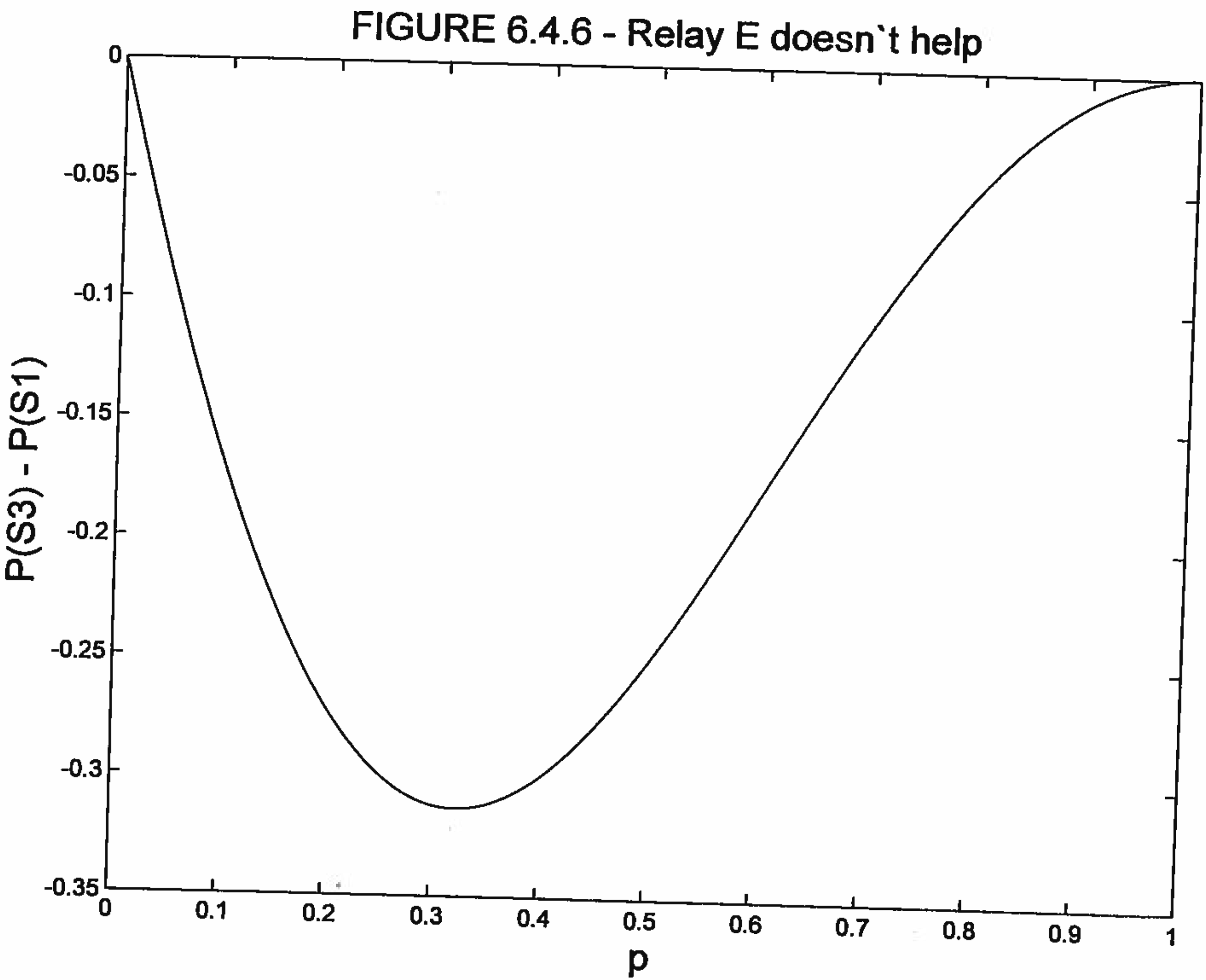
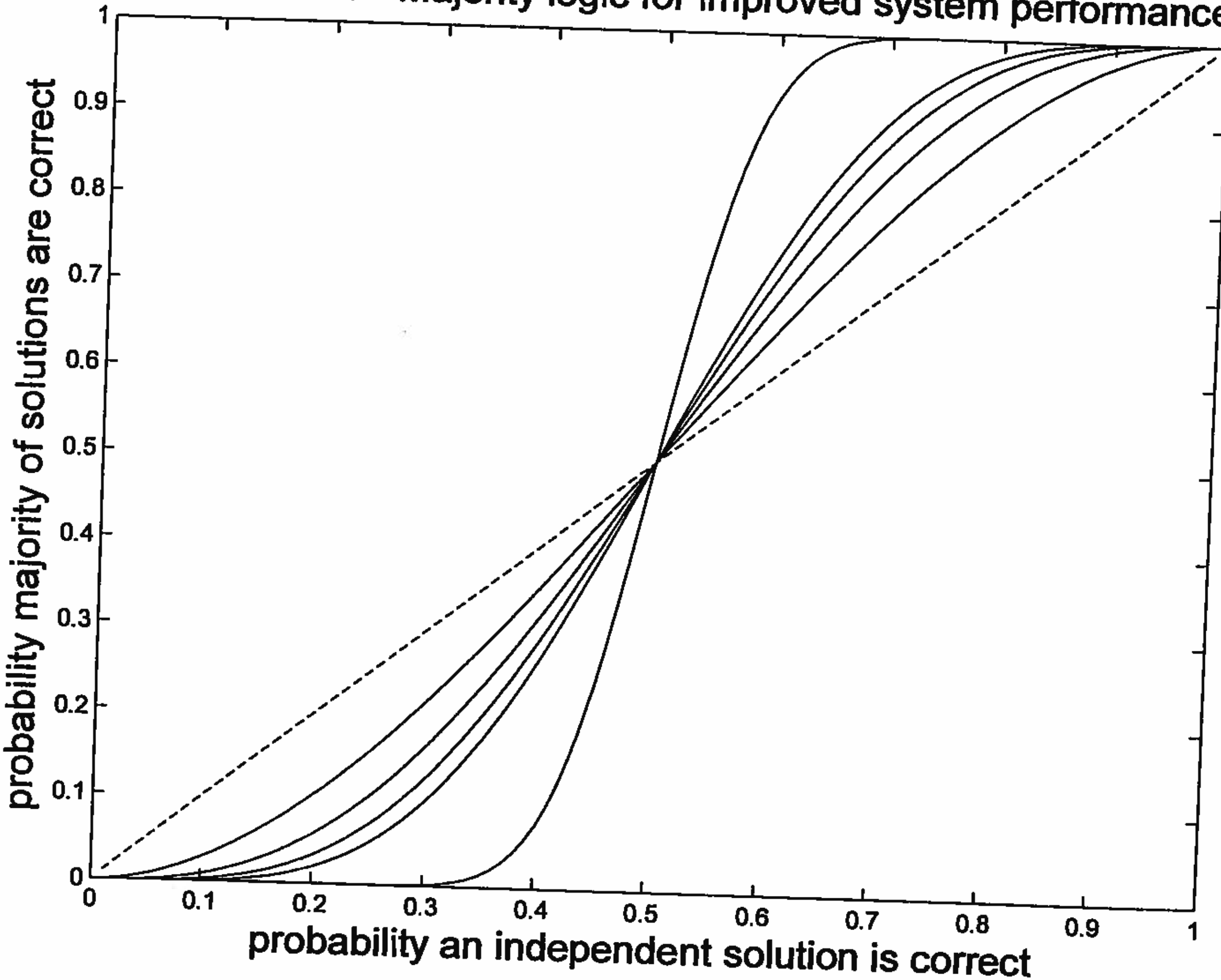


FIGURE 6.4.6 - Relay E doesn`t help
P(S3) - P(S1)
p
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$$(6.5.1) \qquad P(n,p) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} p^{k}(1-p)^{n-k},$$

FIGURE 6.5.1 - Majority logic for improved system performance



$$H = -[p \log_2(p) + (1-p)\log_2(1-p)] \tag{7.1.1}$$

$$H = -\sum_{i=1}^{n} p_i \log_2(p_i), \quad \sum_{i=1}^{n} p_i = 1, \; p_i \geq 0. \tag{7.1.2}$$

$$C = \lim_{T \to \infty} \frac{\log_2\{N(T)\}}{T} \text{ bits/unit time,} \tag{7.1.3}$$

$$(7.1.4) \qquad N(T) = N(T-2) + N(T-3) + N(T-4).$$

$$N(T) = Ka^T \tag{7.1.5}$$

$$a^4 - a^2 - a - 1 = 0. \tag{7.1.6}$$

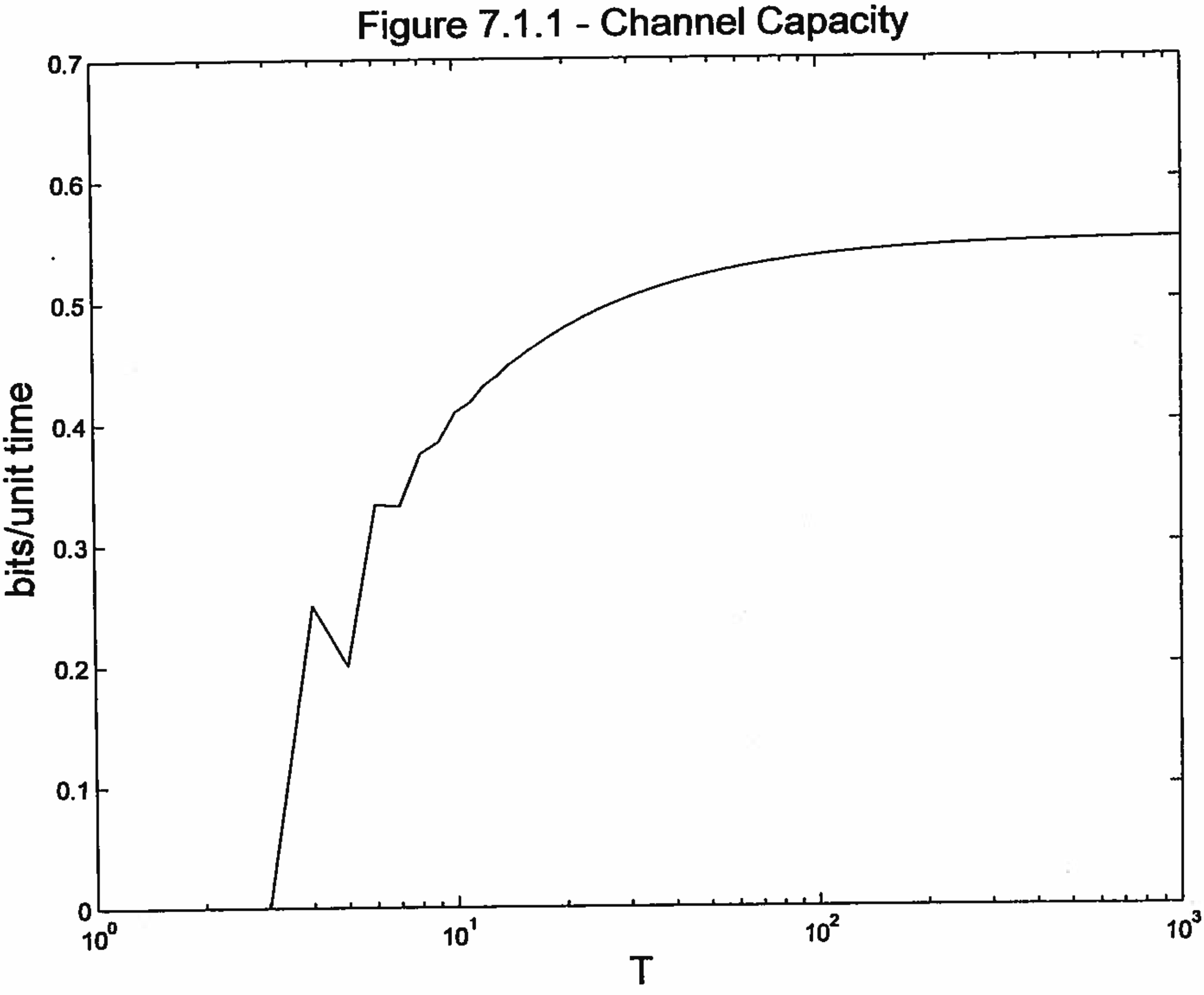


Figure 7.1.1 - Channel Capacity
bits/unit time
T
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10^0
10^1
10^2
10^3

alue of K). That is, the general solution

$$N(T) = K_1(-1)^T + K_2(1.46557)^T + K_3(-0.23278 + i\,0.79255)^T + K_4(-0.23278 - i\,0.79255)^T. \tag{7.1.7}$$

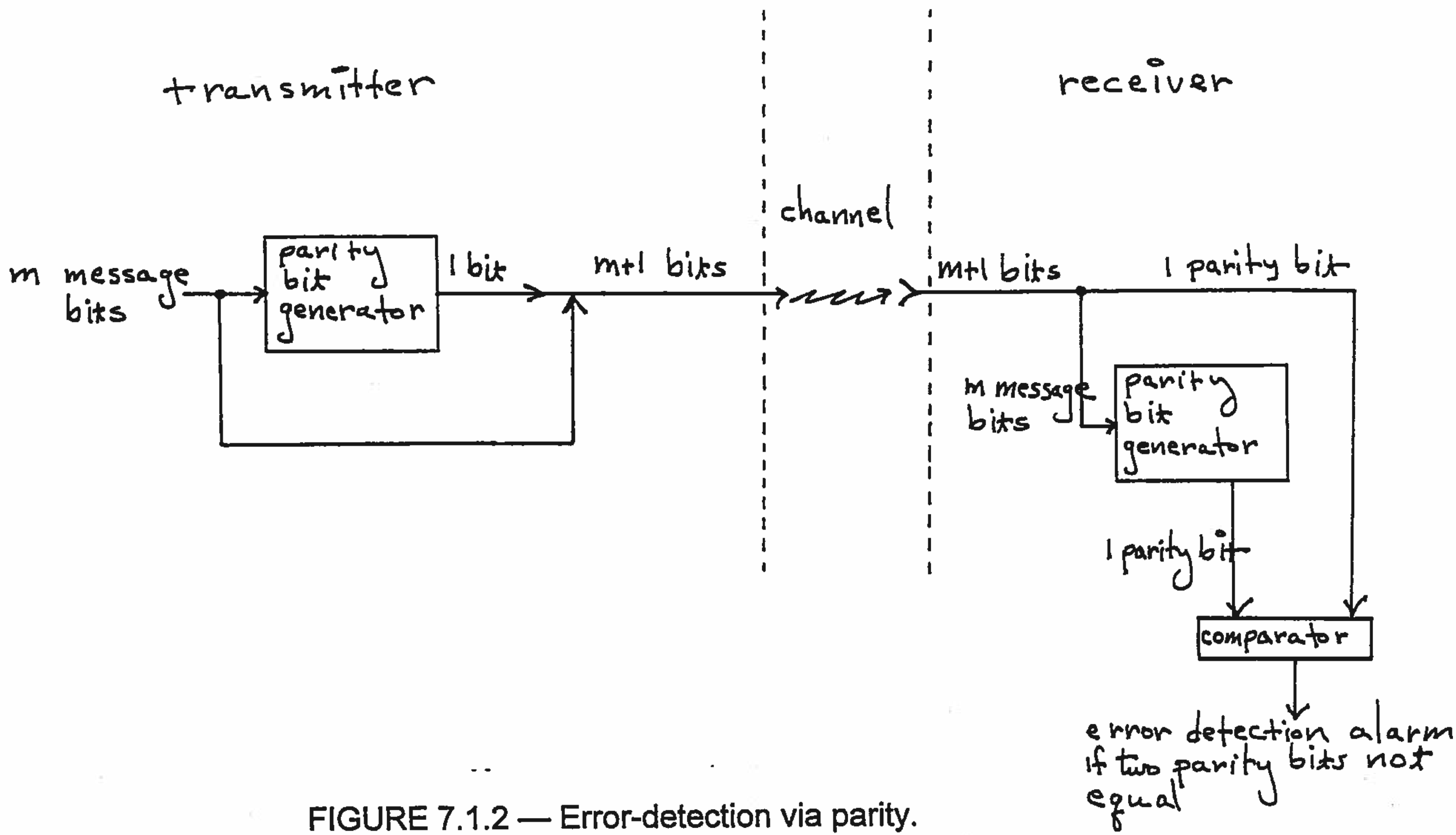


FIGURE 7.1.2 — Error-detection via parity.

$$\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 1 \end{array}$$

(that is, 1 + 4 = 5 in decimal) in which no carry is produced in any of the bit positions, while

$$\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}$$

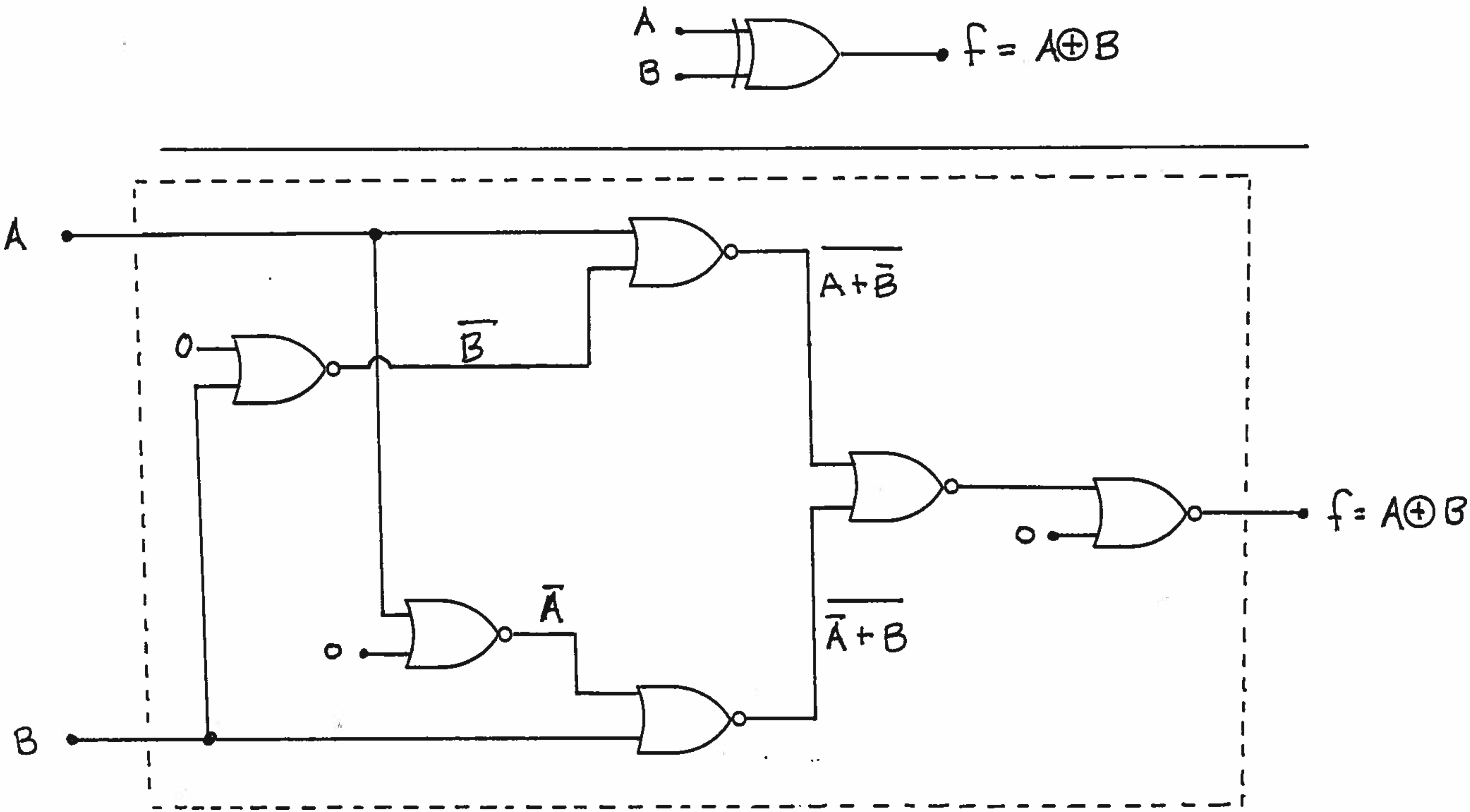


FIGURE 7.2.1 — The XOR logic gate and how to build it with NORs

$$\begin{array}{cccc} & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & - & - & - \\ 1 & 1 & 1 & 0 \end{array}$$

A	B	S	C_O
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Thus,

(7.2.1) $$S = \bar{A}B + A\bar{B} = A \oplus B$$

and

(7.2.2) $$C_O = AB.$$

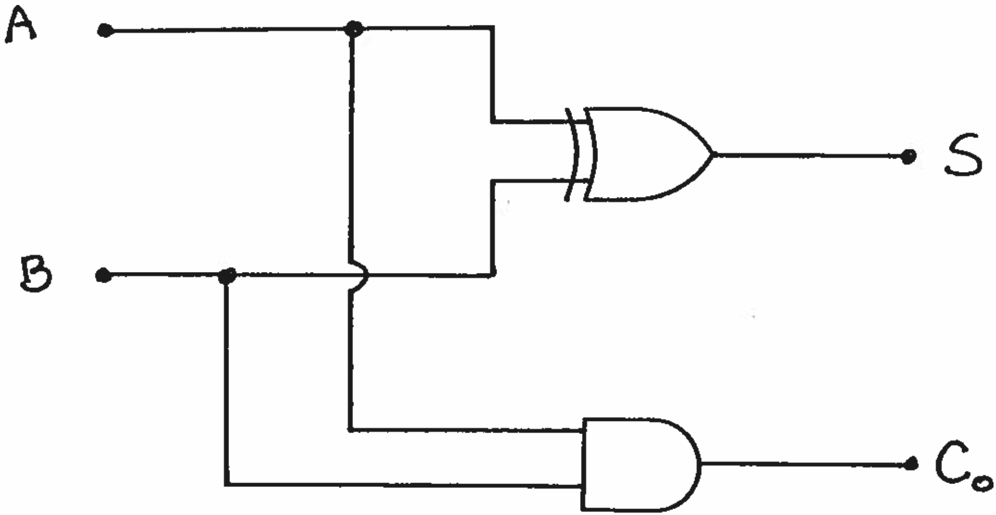


FIGURE 7.2.2 — The half-adder

A	B	C_j	S	Co
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$$C_o = (A \oplus B)C_i + AB. \quad (7.2.3)$$

$$S = (\bar{A}\bar{B} + AB)C_i + (A \oplus B)\bar{C}_i. \tag{7.2.4}$$

$$(7.2.5) \qquad S = (\overline{A \oplus B})C_i + (A \oplus B)\bar{C}_i.$$

$$S = (A \oplus B) \oplus C_i. \tag{7.2.6}$$

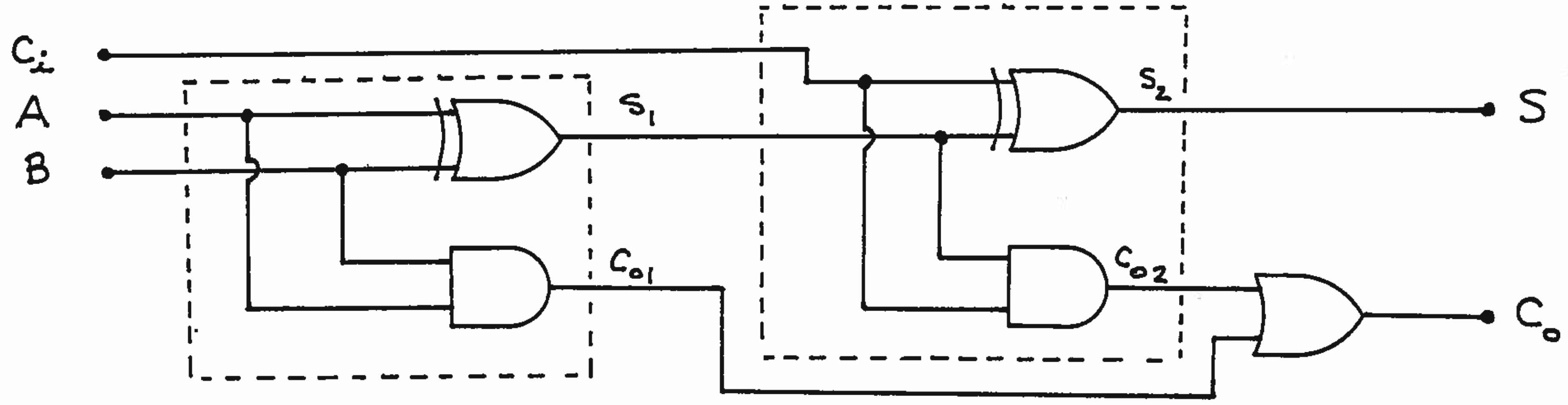
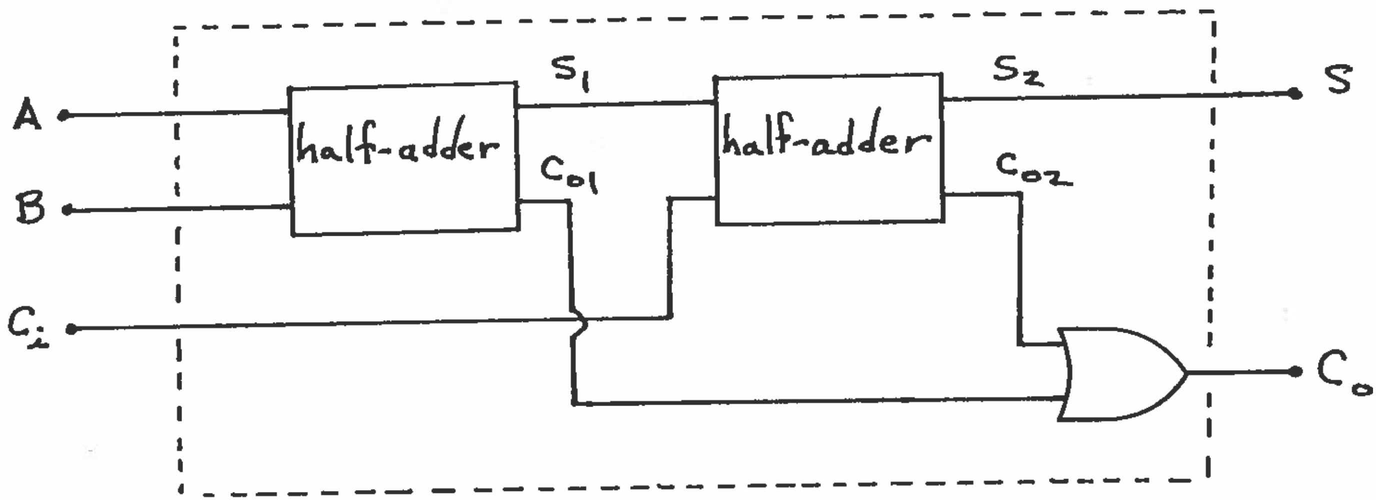


FIGURE 7.2.3 — The full-adder (part 1)



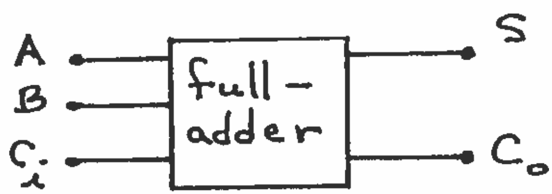


FIGURE 7.2.4 — The full-adder (part 2)

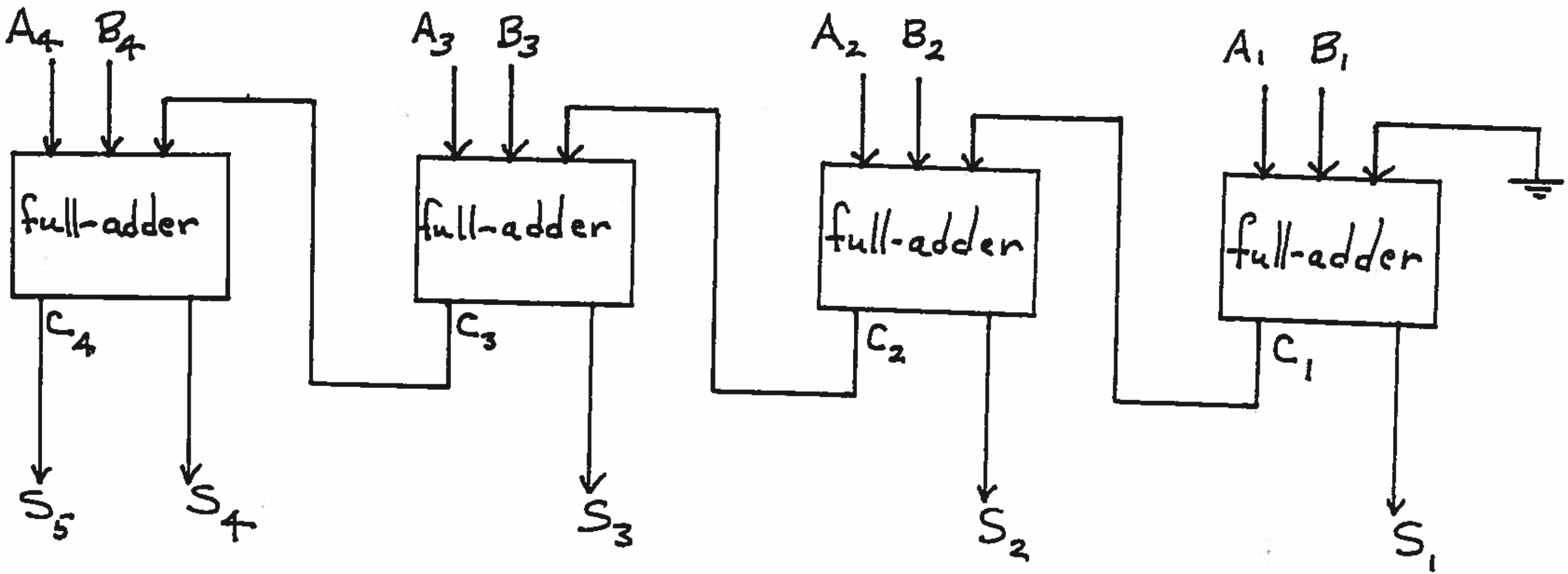


FIGURE 7.2.5 — Adding two, 4-bit numbers

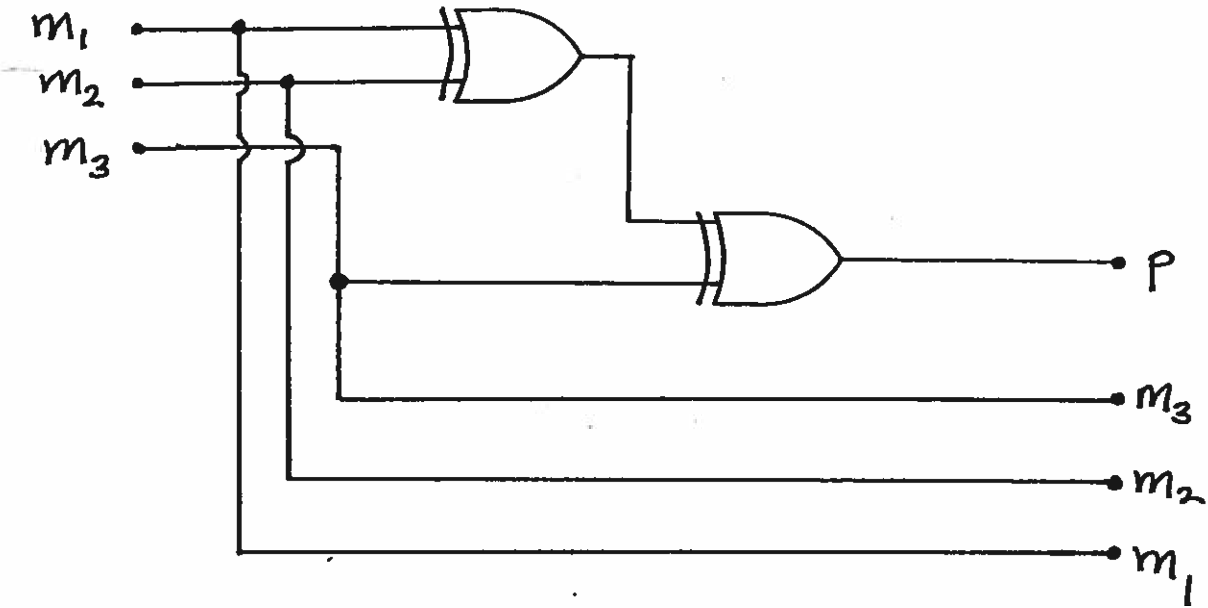


FIGURE 7.3.1 — A parity-generator circuit (at source)

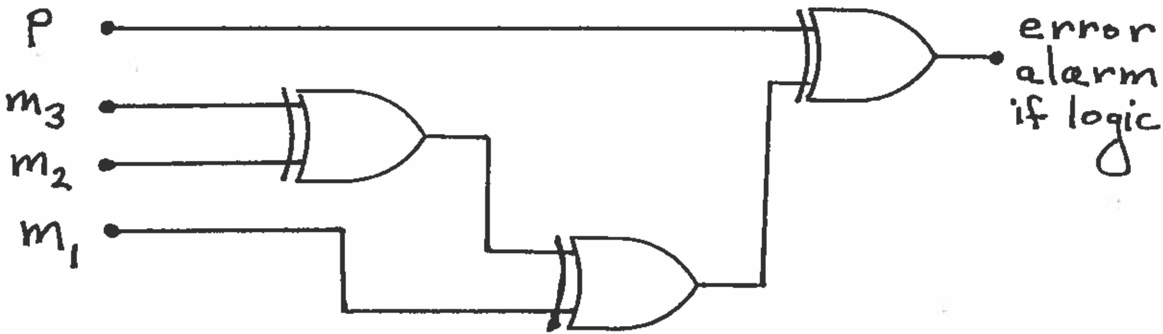


FIGURE 7.3.2 — A parity-checking circuit (at receiver)

$$d \geq 2t + 1. \tag{7.4.1}$$

$$(7.4.2) \qquad S(n,t) = \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{t}$$

$$(7.4.3) \qquad M \leq \frac{2^n}{1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{t}}.$$

$$2^k \geq m + k + 1. \tag{7.5.1}$$

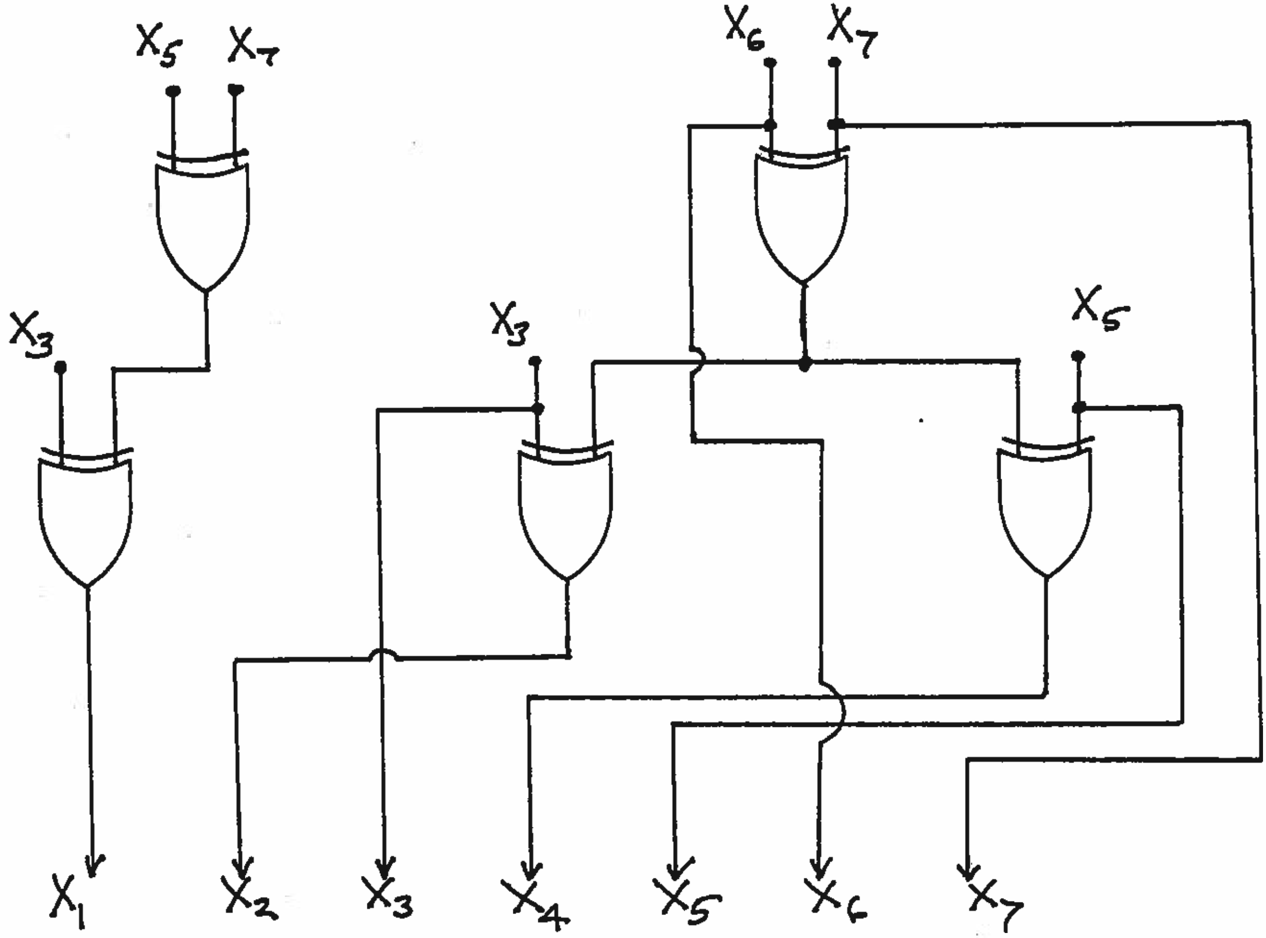


FIGURE 7.5.1 — Single-error correcting Hamming encoder

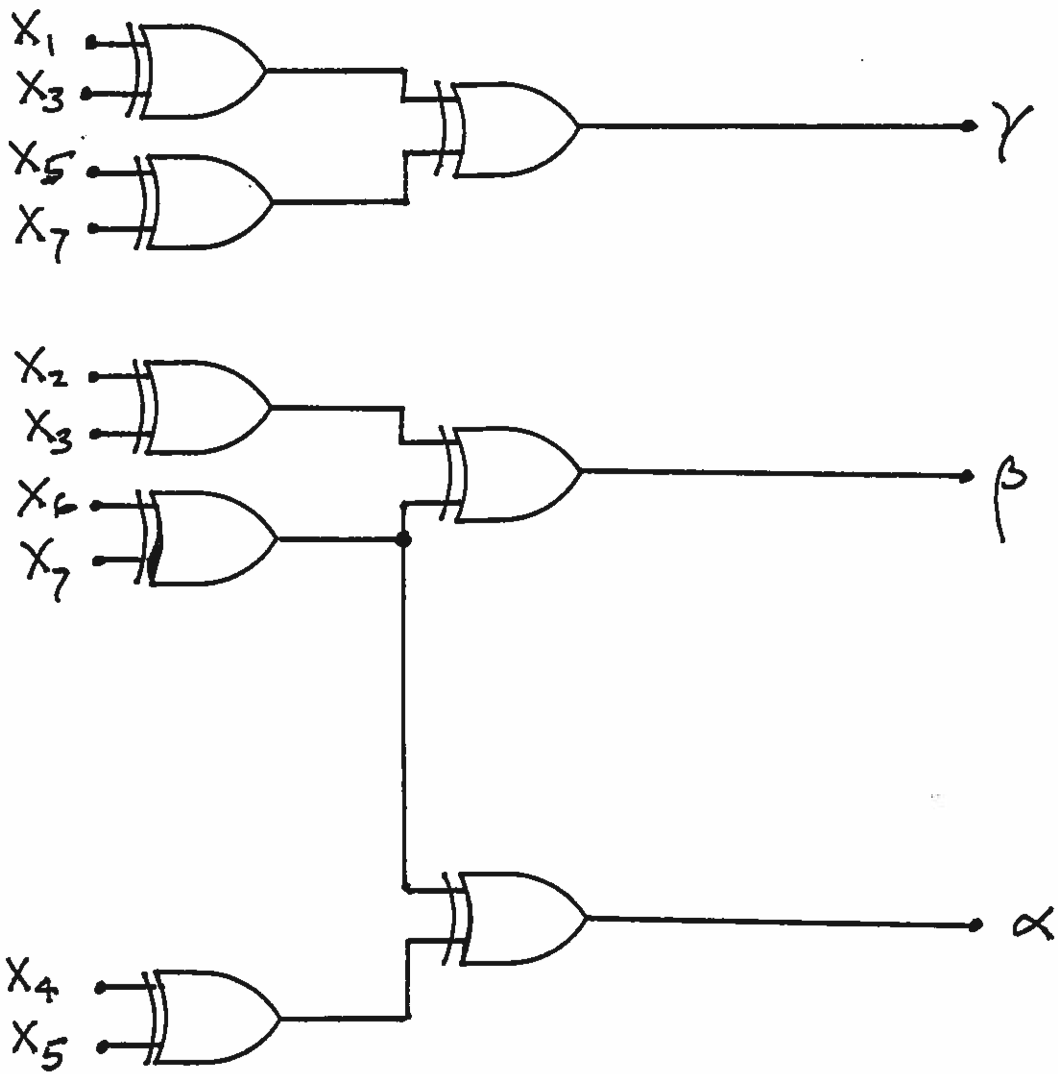


FIGURE 7.5.2 — Syndrome generator (at receiver)

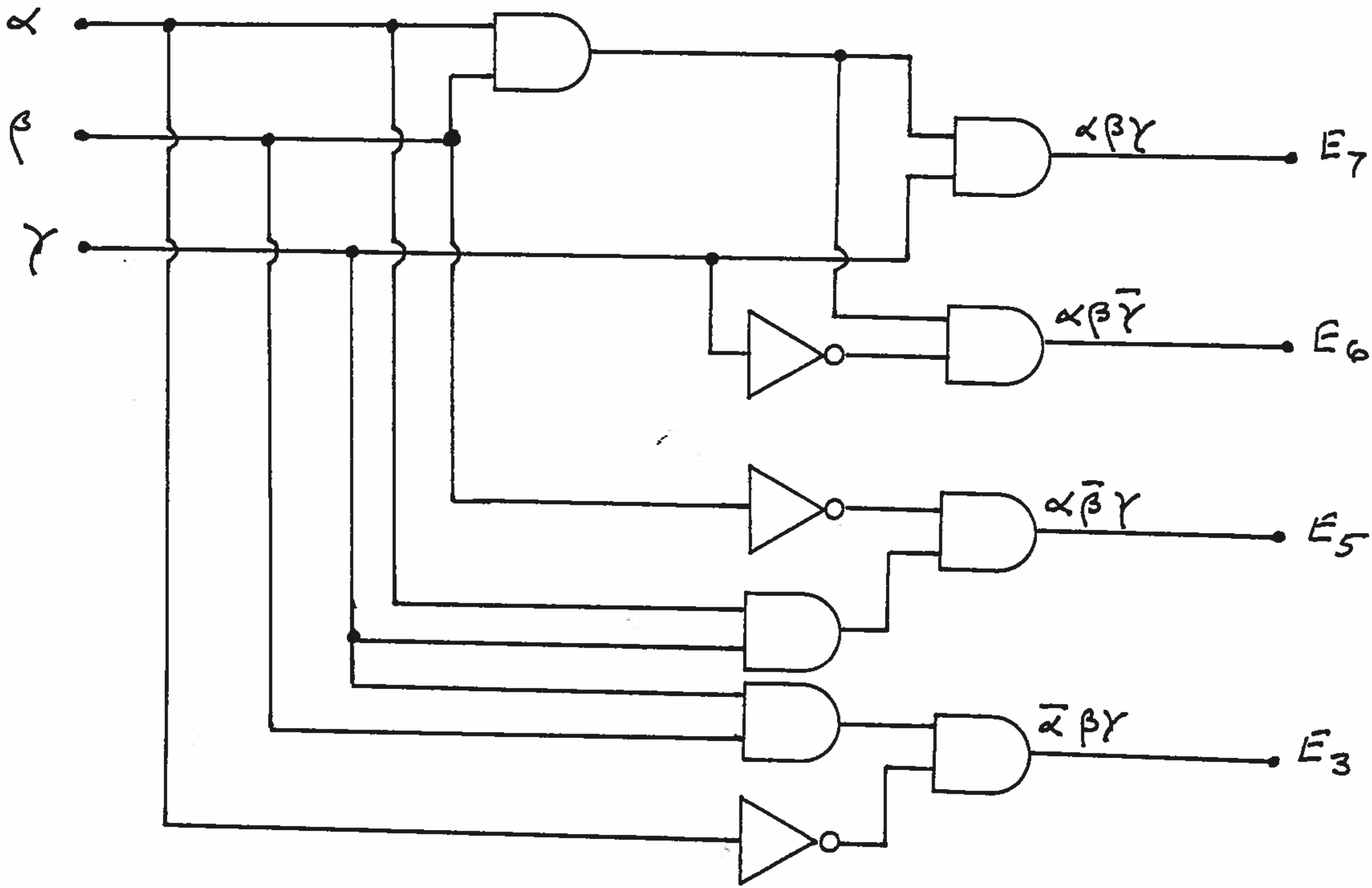


FIGURE 7.5.3 — Syndrome decoder (at receiver)

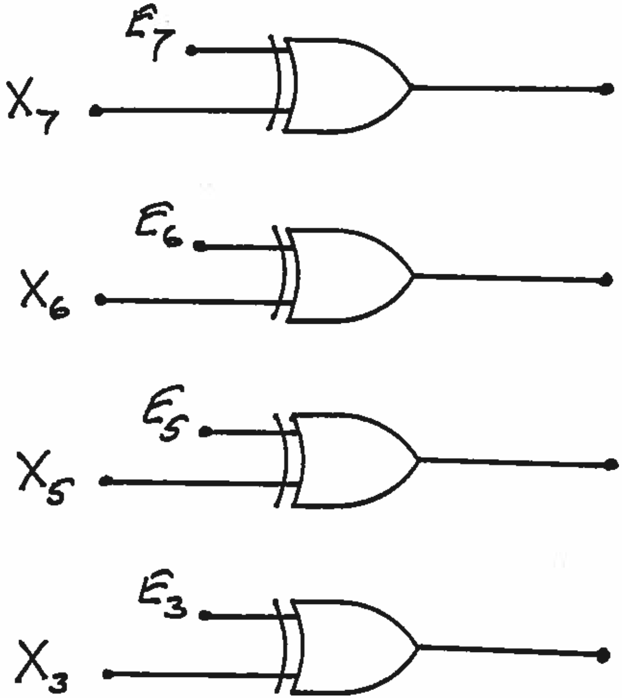


FIGURE 7.5.4 — Corrected (if necessary) message bits (at receiver)

[(A,A,C,C,B), ()]

[(A,A), (C,C,B)]

[(A,A,C,B), (C)]

[(A,C), (A,C,B)]

[(A,C,C,B), (A)]

[(A), (A,C,C,B)]

[(A,C,B), (A,C)]

[(C), (A,A,C,B)]

[(C,C,B), (A,A)]

[(), (A,A,C,C,B)].

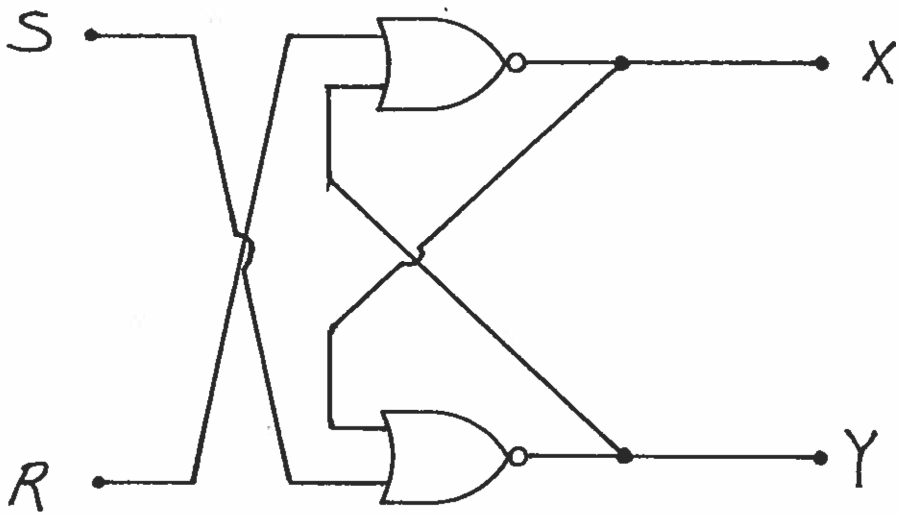


FIGURE 8.2.1 — The NOR bistable latch

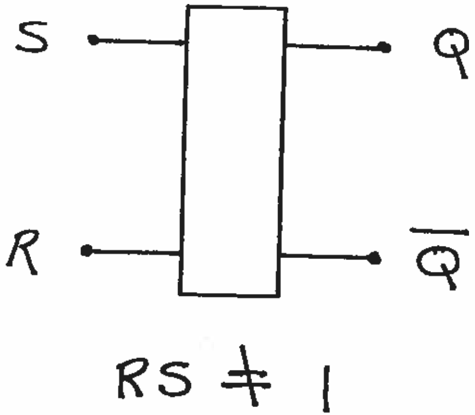


FIGURE 8.2.2 — The RS latch

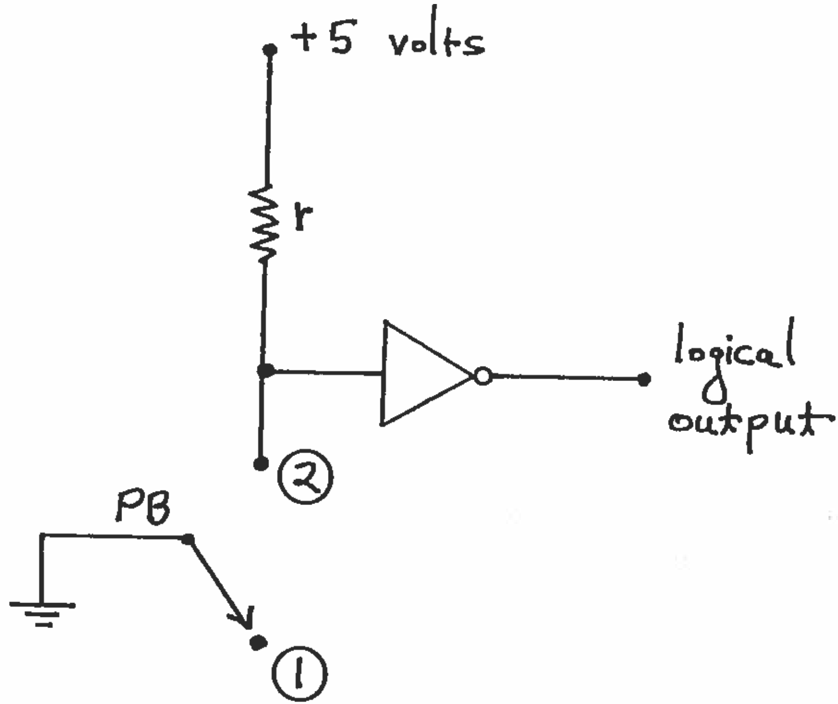


FIGURE 8.3.1 — One way to generate the start signal for a digital machine

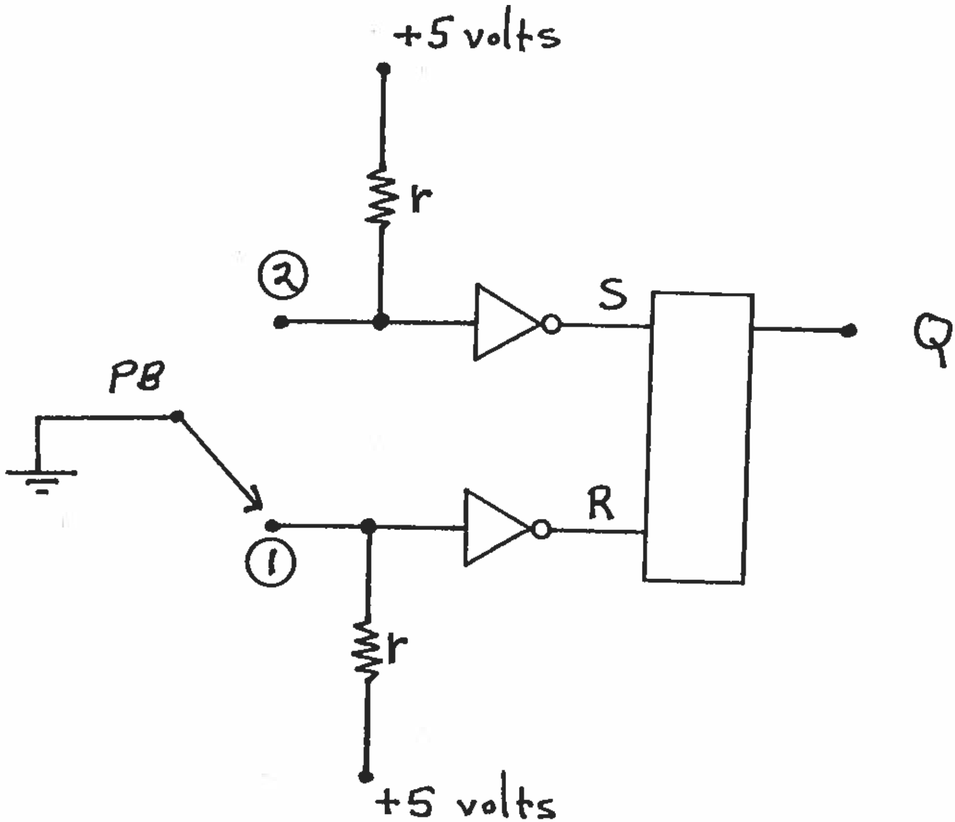


FIGURE 8.3.2 — Switch debouncing with the RS latch

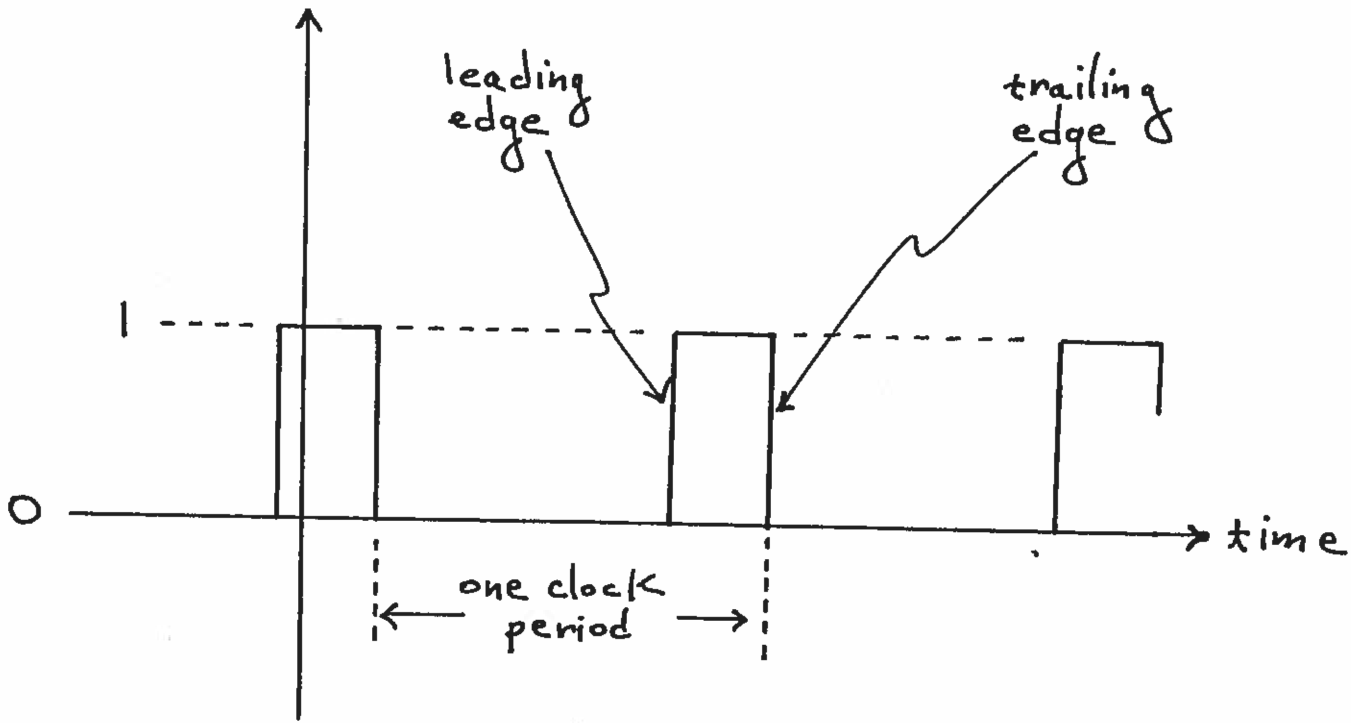


FIGURE 8.3.3 — A clock signal

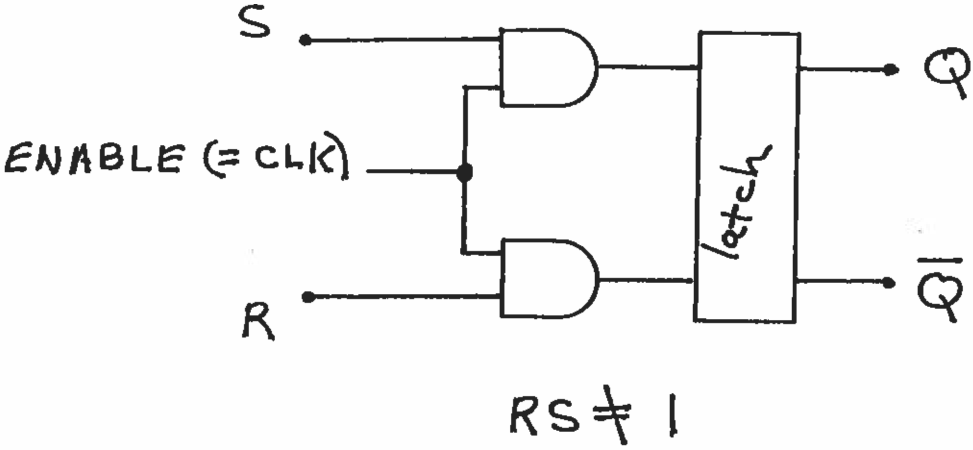


FIGURE 8.3.4 — The RS flip-flop

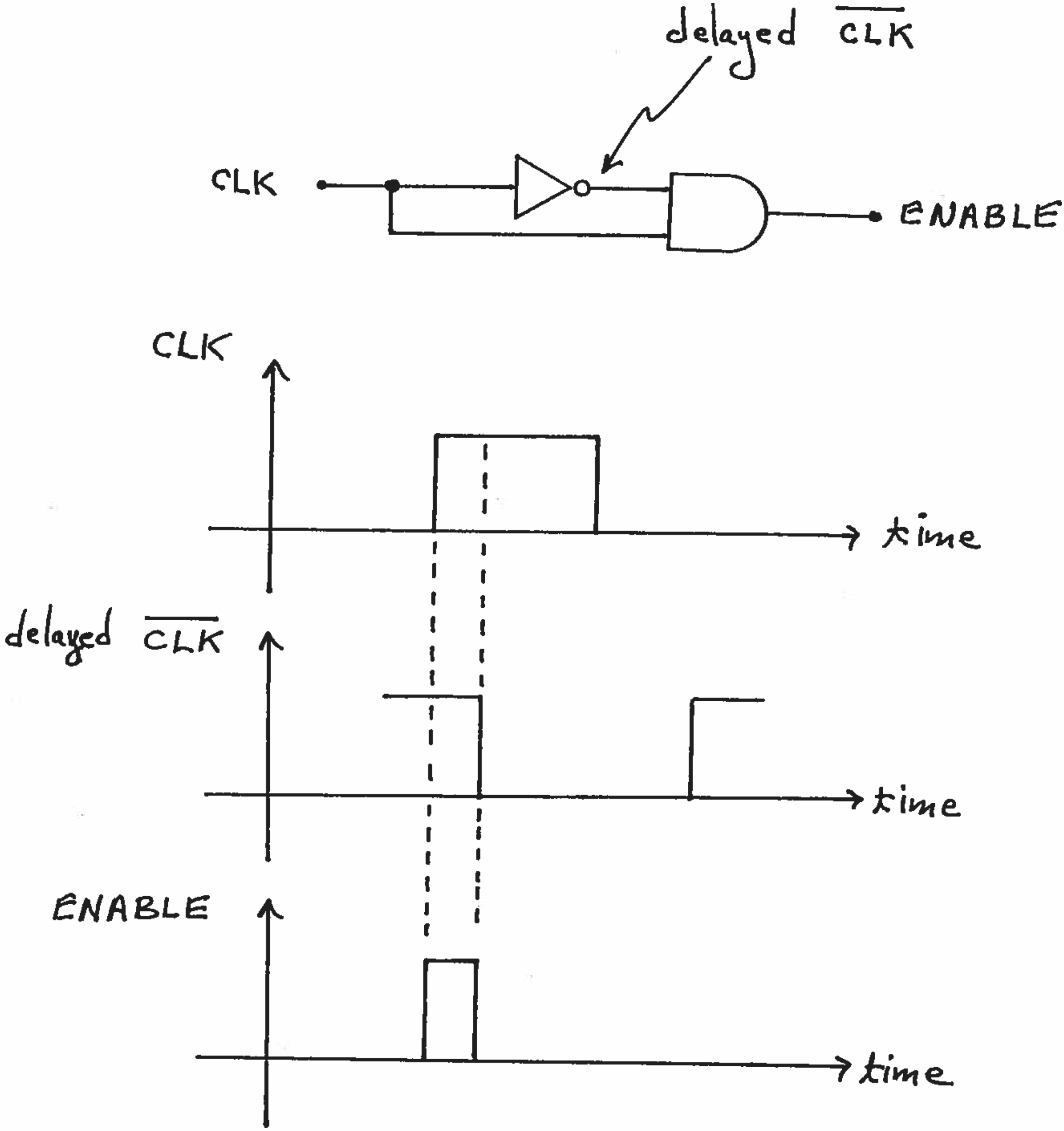


FIGURE 8.3.5 Leading-edge detector

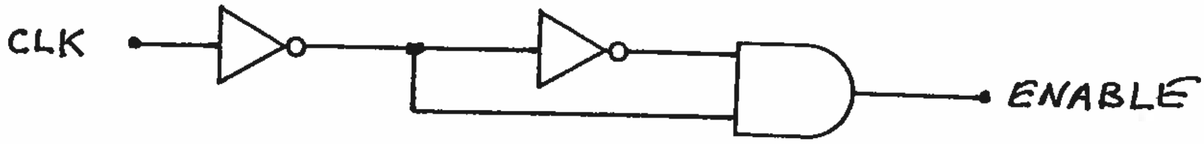


FIGURE 8.3.6 — Trailing-edge detector

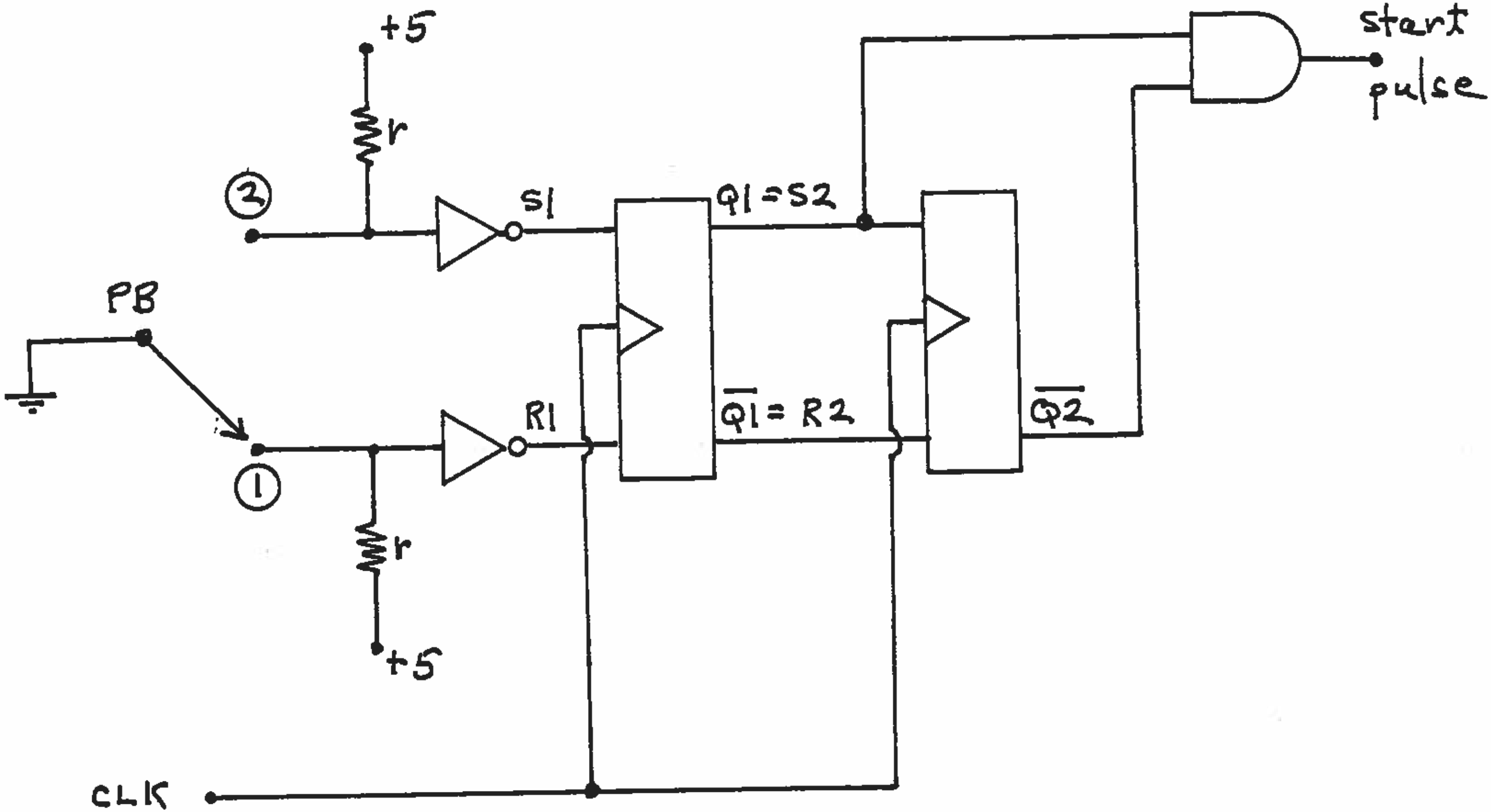


FIGURE 8.3.7 — How to generate the start signal for a digital machine

$T^{(n)}$	$Q^{(n)}$	$Q^{(n+1)}$	$R^{(n)}$	$S^{(n)}$
0	0	0	0,1	0
1	0	1	0	1
0	1	1	0	0,1
1	1	0	1	0

$T^{(n)}$ and $Q^{(n)}$:

$$R^{(n)} = T^{(n)}Q^{(n)} + \left(\overline{T^{(n)}}\ \overline{Q^{(n)}}\right) \tag{8.4.1}$$

and

$$S^{(n)} = T^{(n)}\ \overline{Q^{(n)}} + \left(\overline{T^{(n)}}\ Q^{(n)}\right). \tag{8.4.2}$$

$D^{(n)}$	$Q^{(n)}$	$Q^{(n+1)}$	$R^{(n)}$	$S^{(n)}$
0	0	0	0,1	0
1	0	1	0	1
0	1	0	1	0
1	1	1	0	0,1

i). Writing the

$$R^{(n)} = \overline{D^{(n)}}Q^{(n)} + \left(\overline{D^{(n)}}\;\overline{Q^{(n)}}\right) \tag{8.4.3}$$

and

$$S^{(n)} = D^{(n)}\,\overline{Q^{(n)}} + \left(D^{(n)}Q^{(n)}\right). \tag{8.4.4}$$

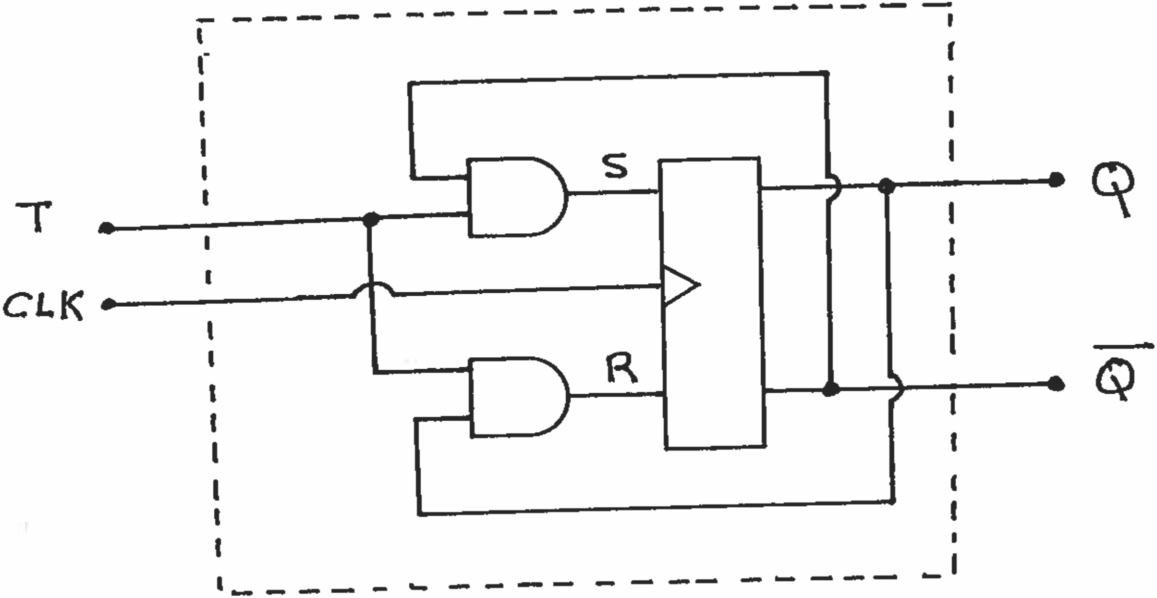


FIGURE 8.4.1 — The T flip-flop

$$(8.4.5) \qquad R^{(n)} = \overline{D^{(n)}}$$

and

$$(8.4.6) \qquad S^{(n)} = D^{(n)}.$$

machine state at clock n	$X^{(n)}$	$Y^{(n)}$	$Z^{(n)}$	machine state at clock n+1
1	0	1	1	2
1	1	1	1	0
2	0	1	0	2
2	1	0	1	3
3	0	1	0	3
3	1	1	0	1

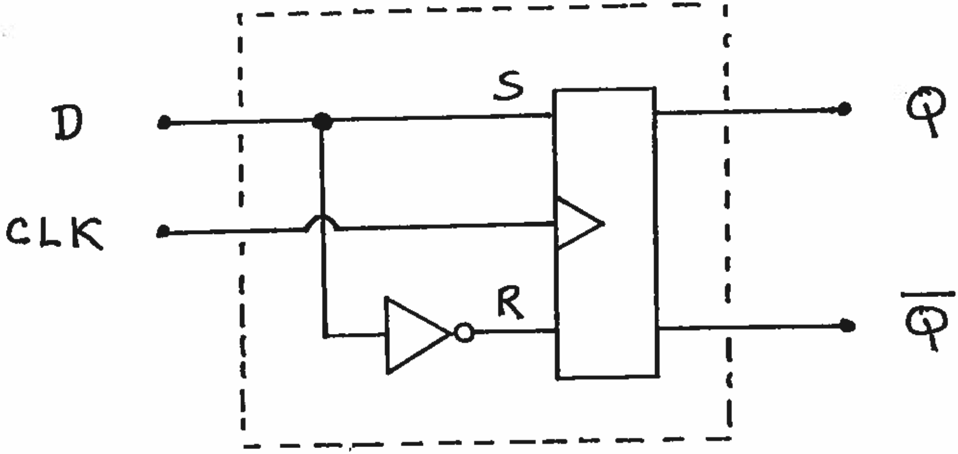


FIGURE 8.4.2 — The D flip-flop

$Q1^{(n)}$	$Q2^{(n)}$	X	$Q1^{(n+1)}$	$Q2^{(n+1)}$	$T1^{(n)}$	$T2^{(n)}$	Y	Z
0	1	0	1	0	1	1	1	1
0	1	1	0	0	0	1	1	1
1	0	0	1	0	0	0	1	0
1	0	1	1	1	0	1	0	1
1	1	0	1	1	0	0	1	0
1	1	1	0	1	1	0	1	0

From this table we can write the T1 and T2 equations as

(8.5.1) $$T1^{(n)} = \overline{Q1^{(n)}}\, Q2^{(n)}\, \overline{X^{(n)}} + Q1^{(n)}\, Q2^{(n)}\, X^{(n)}$$

and

$$T2^{(n)} = \overline{Q1^{(n)}}\, Q2^{(n)}\, \overline{X^{(n)}} + \overline{Q1^{(n)}}\, Q2^{(n)}\, X^{(n)} + Q1^{(n)}\, \overline{Q2^{(n)}}\, X^{(n)}$$

or,

(8.5.2) $$T2^{(n)} = \overline{Q1^{(n)}}\, Q2^{(n)} + Q1^{(n)}\, \overline{Q2^{(n)}}\, X^{(n)}.$$

Also, the output equations are

(8.5.3) $\overline{Y^{(n)}} = Q1^{(n)}\, \overline{Q2^{(n)}}\, X^{(n)}$ (that is, $Y^{(n)} = \overline{Q1^{(n)}\, \overline{Q2^{(n)}}\, X^{(n)}}$

and, as the table shows by inspection,

(8.5.4) $$Z^{(n)} = T2^{(n)}.$$

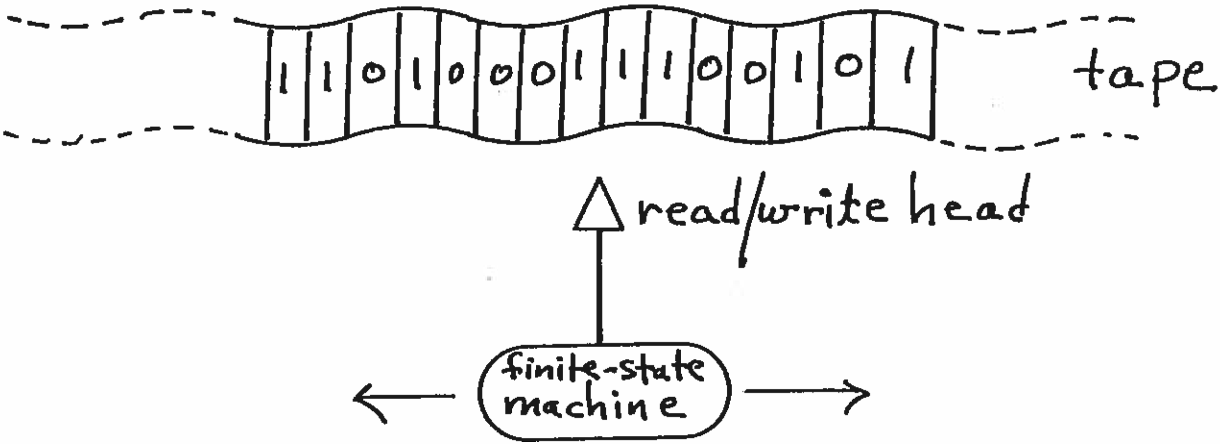


FIGURE 9.1.1 — A Turing machine

present state	tape symbol	operation	next state
1	0	0/R	1
1	1	0/-	2
2	0	0/R	2
2	1	0/-	3
3	0	0/R	3
3	1	1/R	4
4	0	1/-	5
4	1	1/R	4
5	0	halt	0
5	1	1/R	5

$0 + 2 = \cdots 0101110 \cdots$ which should give $\cdots 01110 \cdots$

$2 + 0 = \cdots 0111010 \cdots$ which should give $\cdots 01110 \cdots$

$2 + 3 = \cdots 0111011110 \cdots$ which should give $\cdots 01111110 \cdots$.

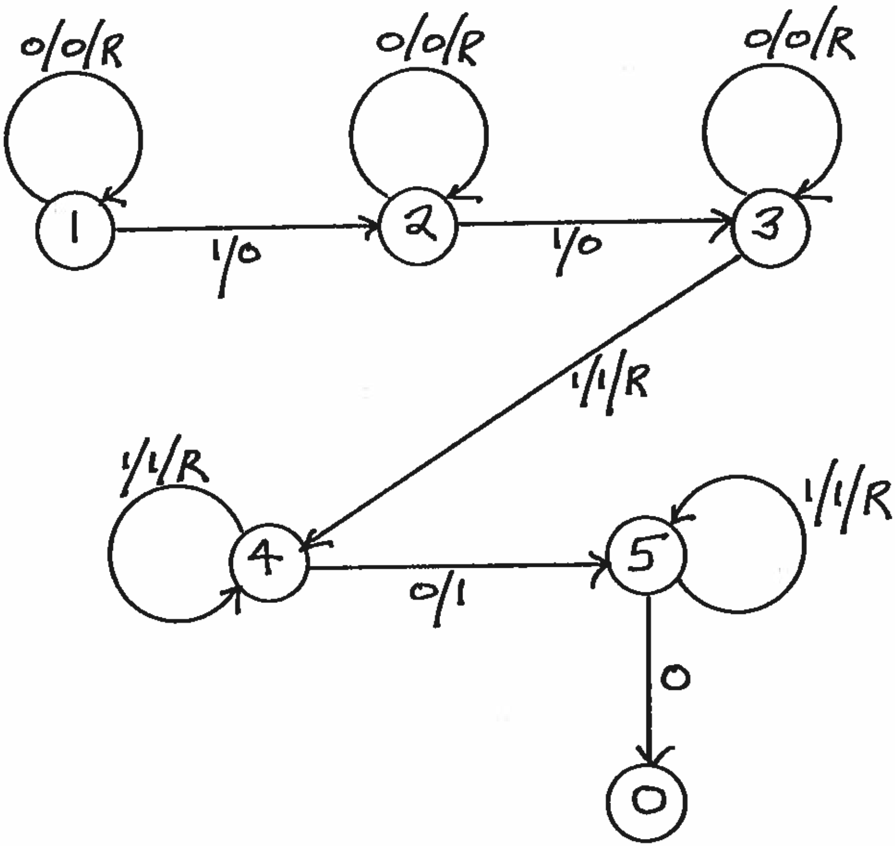


FIGURE 9.2.1 — The state-transition diagram for a Turing machine adder

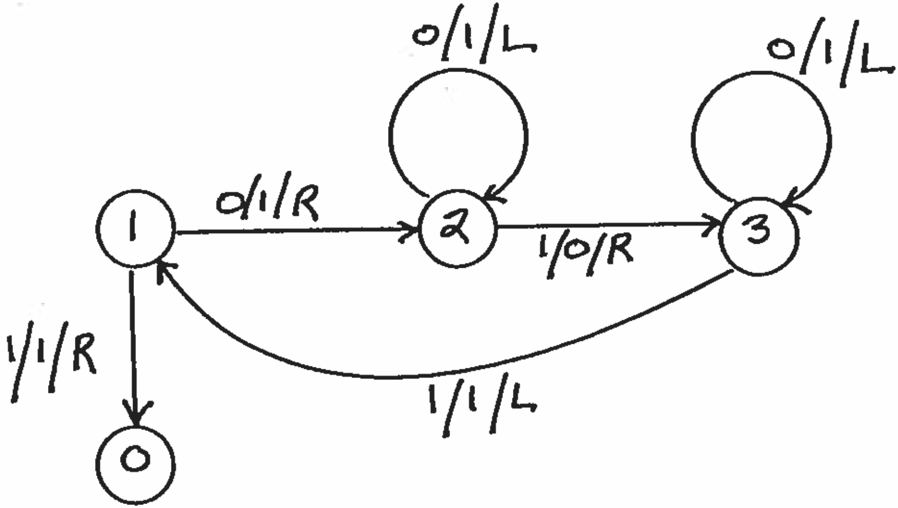


FIGURE 9.2.2 — Radó's Busy Beaver Turing machine

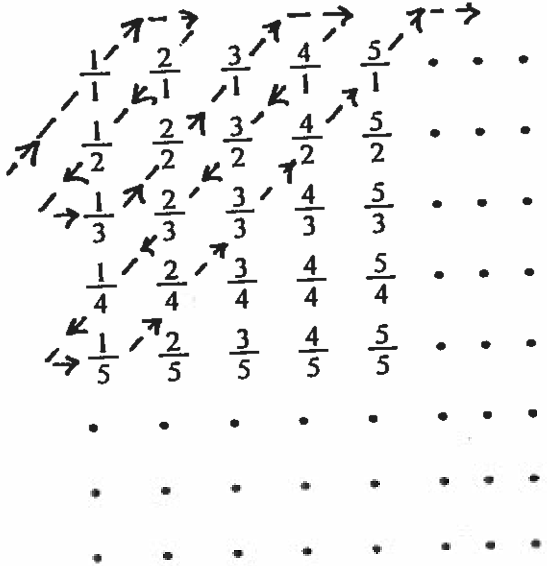


FIGURE 9.3.1 — Cantor's infinite matrix of the rationals

$$
\begin{array}{llllllll}
d_1 = & 0. & d_{11} & d_{12} & d_{13} & d_{14} & \cdots \\
d_2 = & 0. & d_{21} & d_{22} & d_{23} & d_{24} & \cdots \\
d_3 = & 0. & d_{31} & d_{32} & d_{33} & d_{34} & \cdots \\
d_4 = & 0. & d_{41} & d_{42} & d_{43} & d_{44} & \cdots \\
\cdot & \cdot & & & & & \cdots \\
\cdot & \cdot & & & & & \cdots \\
\cdot & \cdot & & & & & \cdots
\end{array}
$$

rado.m

```
state=1;tape=zeros(1,50);location=25;shift=0;
while state>0
    symbol=tape(location);
    if state==1
        if symbol==0
            tape(location)=1;location=location+1;state=2;
        else
            tape(location)=1;location=location+1;state=0;
        end
    elseif state==2
        if symbol==0
            tape(location)=1;location=location-1;state=2;
        else
            tape(location)=0;location=location+1;state=3;
        end
    else
        if symbol==0
            tape(location)=1;location=location-1;state=3;
        else
            tape(location)=1;location=location-1;state=1;
        end
    end
    shift=shift+1;
end
sum(tape),shift
```

A	B	F
0	0	1
0	1	1
1	0	1
1	1	0

A	B	C	A	B	C'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

$$
\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

then

$$
\mathbf{T} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{10.3.1}
$$

A	B	C	A	B'	C'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then

$$(10.3.2) \qquad F \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

system is written as

$$(10.5.1)\qquad |\psi\rangle = c_1|0\rangle + c_2|1\rangle = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}c_1\\0\end{bmatrix} + \begin{bmatrix}0\\c_2\end{bmatrix}$$

$$= \begin{bmatrix}c_1\\c_2\end{bmatrix},$$

$$\sum_{k=1}^{2^n} |c_k|^2 = 1.$$

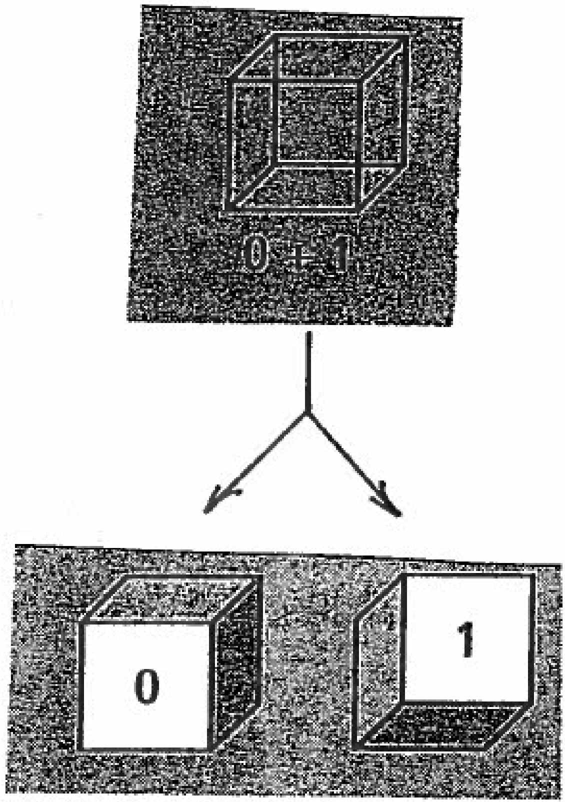


FIGURE 10.5.1 — An optical illusion illustrating both state superposition and measurement collapse

$$|\psi'\rangle = c_1\,|1\rangle + c_2\,|0\rangle. \tag{10.6.1}$$

$$|\psi'\rangle = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}. \tag{10.6.2}$$

If we represent the quantum inverter gate by the symbol **N**, then **N** 'operating on' $|\psi\rangle$ should give $|\psi'\rangle$. That is,

$$\mathbf{N}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}. \qquad (10.6.3)$$

What 'operating on' a two-element column vector gives another two-element column vector? A 2-by-2 matrix! That is, if we write

$$\mathbf{N}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix},$$

then performing the matrix multiplication gives us the two equations

$$ac_1 + bc_2 = c_2$$

$$dc_1 + ec_2 = c_1$$

which, by inspection, says $a = 0$, $b = 1$ and $d = 1$, $e = 0$. That is, the quantum logic inverter gate is, *mathematically*, the matrix

$$\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (10.6.4)$$

Our result in (10.6.4) makes sense, too, when you ask yourself the question: what should happen to $|\psi\rangle$ if we run it through *two* quantum inverters in series? The answer seems clear: you should get $|\psi\rangle$ back. Do we? Yes, because

$$\mathbf{N}\{\mathbf{N}\,|\,\psi\rangle\} = \mathbf{N}\left\{\mathbf{N}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right\} = \mathbf{N}\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = |\psi\rangle.$$

And, in fact,

$$\mathbf{NN} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

where **I** is the 2-by-2 identity matrix and (of course!) $\mathbf{I}|\psi\rangle = |\psi\rangle$.

$$\mathbf{M}|\psi\rangle = |\psi'\rangle = \begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} ac_1 + bc_2 \\ dc_1 + ec_2 \end{bmatrix}$$

$$= (ac_1 + bc_2)|0\rangle + (dc_1 + ec_2)|1\rangle.$$

Now, let's write $\mathbf{M}^\dagger$ as the *adjoint* of M, which means $\mathbf{M}^\dagger$ is the *conjugated transpose* of **M**. That is,

$$\mathbf{M}^\dagger = \begin{bmatrix} a^* & d^* \\ b^* & e^* \end{bmatrix}.$$

And suppose further that we require $\mathbf{M}^\dagger\mathbf{M} = \mathrm{I}$. That is,

$$\begin{bmatrix} a^* & d^* \\ b^* & e^* \end{bmatrix}\begin{bmatrix} a & b \\ d & e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\begin{gathered} |a|^2 + |d|^2 = 1 \\ ab^* + e^*d = 0 \\ a^*b + d^*e = 0 \\ |b|^2 + |e|^2 = 1 \end{gathered} \qquad (10.6.5)$$

$$\text{(10.6.6)} \qquad \mathbf{H}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\text{(10.6.7)} \qquad \mathbf{H}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

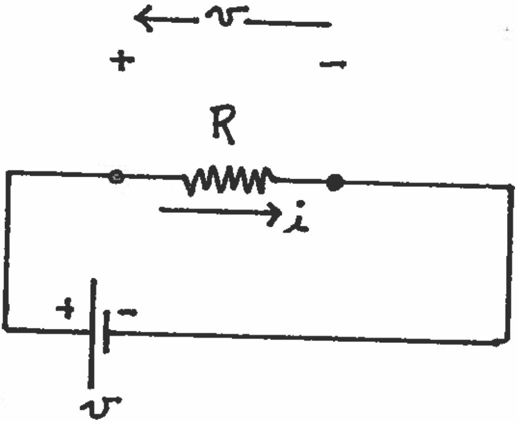


FIGURE A1 – The resistor

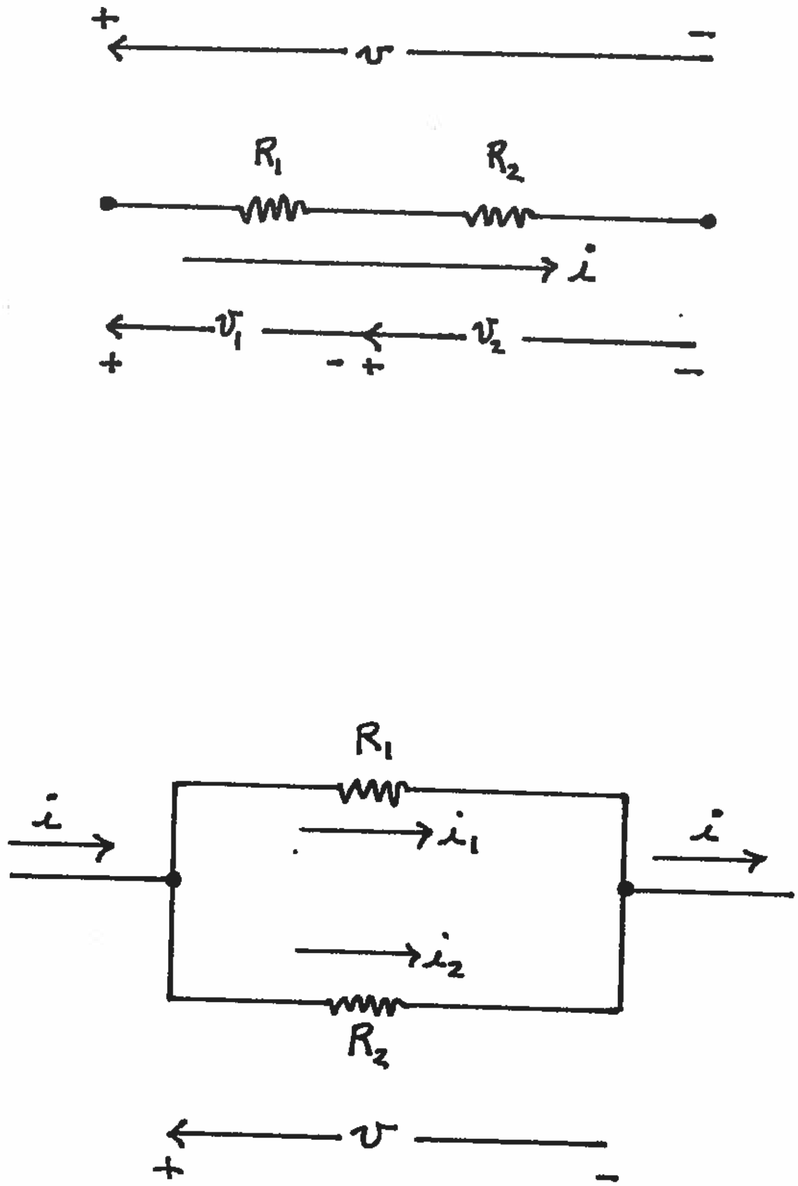


FIGURE A2 – Resistors in series (top) and in parallel (bottom)