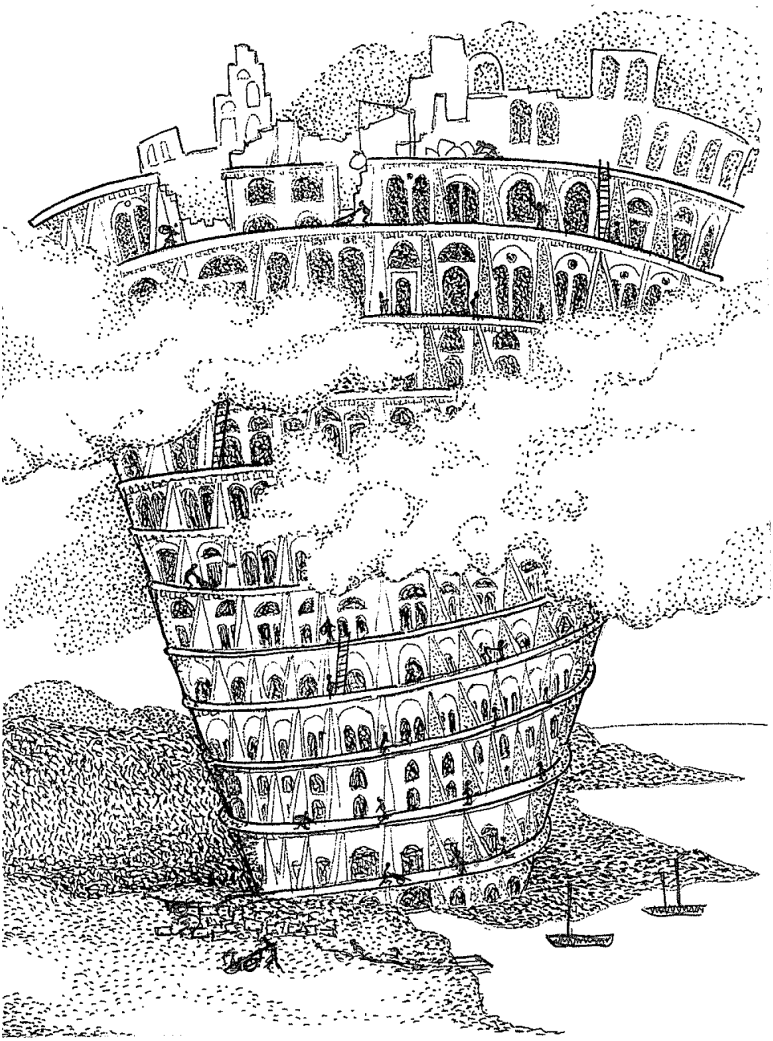
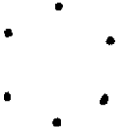
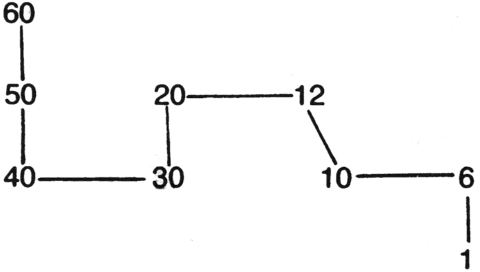
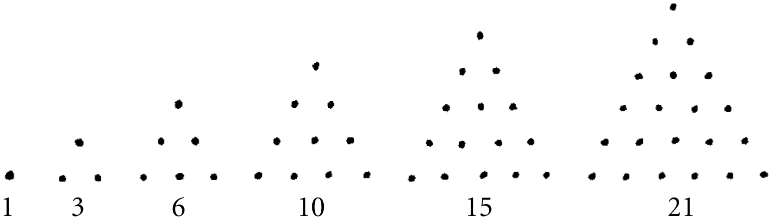


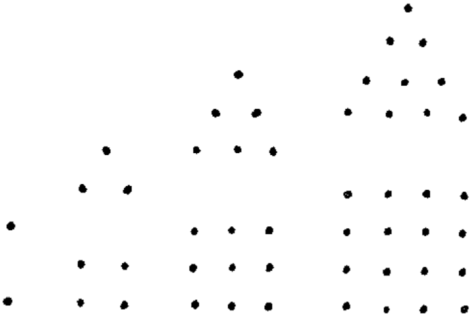
The Tower of Mathematics is the Tower of Babel inverted: its voices grow more coherent as it rises. The image of it is based on Pieter Brueghel's "Little Tower of Babel" (1554).

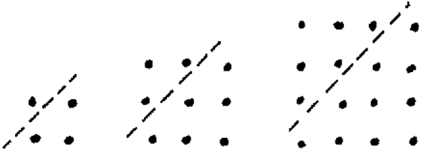
Not only do the names of numbers vary, but, more surprisingly, how we picture them to ourselves. Do you think of "six" as 

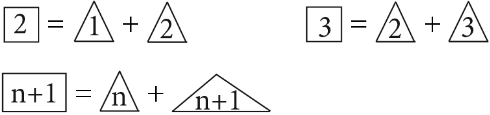
or  or  or  or  ?

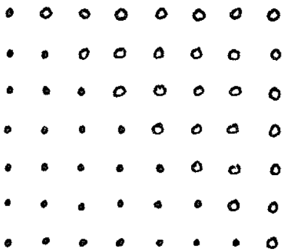

60
50
20
12
40
30
10
6
1


1
3
6
10
15
21

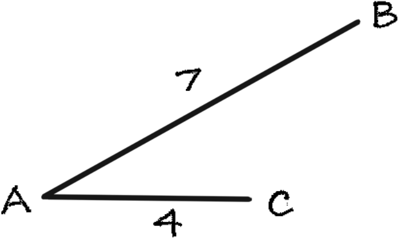




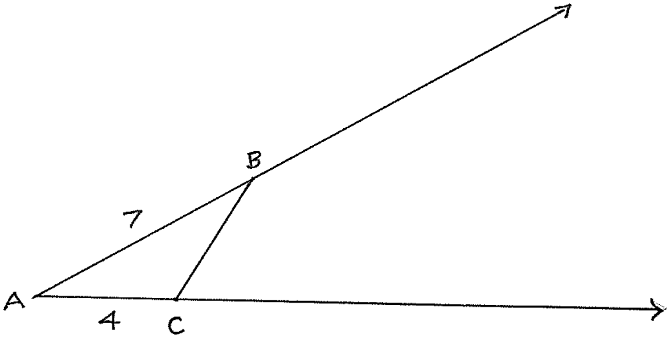

2 = 1 + 2
3 = 2 + 3
n+1 = n + n+1

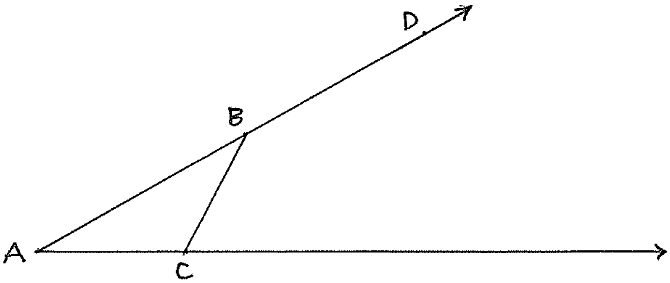


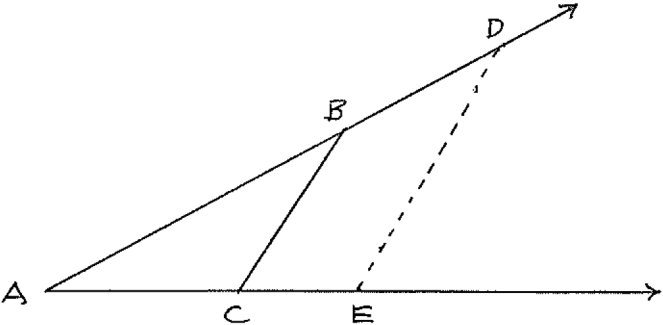
$1 \qquad 1 + 3 \qquad 1 + 3 + 5 \qquad 1 + 3 + 5 + 7.$

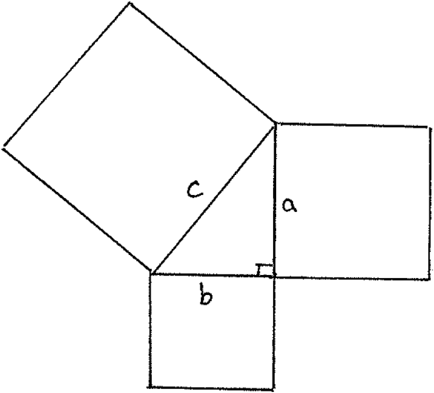


B
7
A
C
4

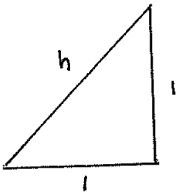

B
7
A
4
C

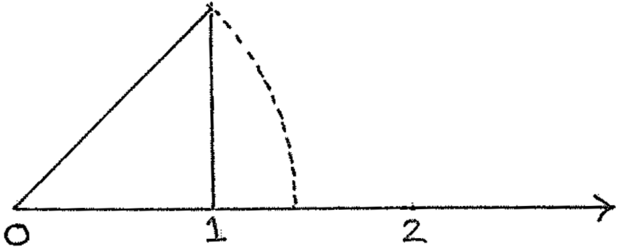

D
B
A
C


D
B
A
C
E

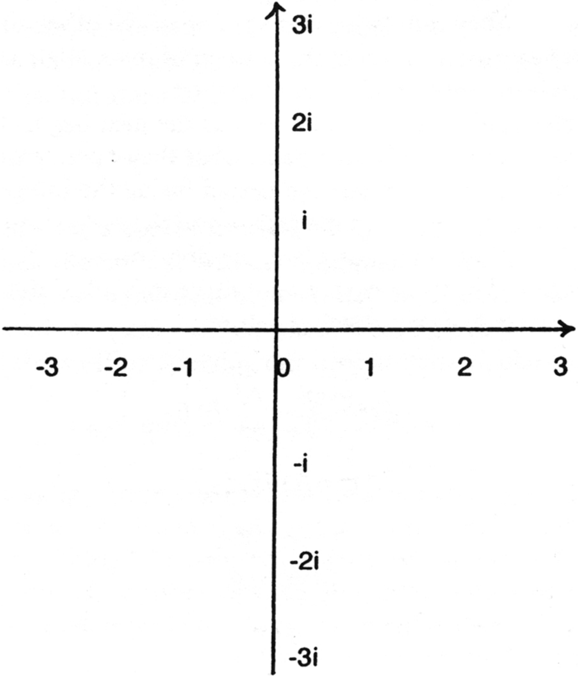


$$a^2 + b^2 = c^2$$

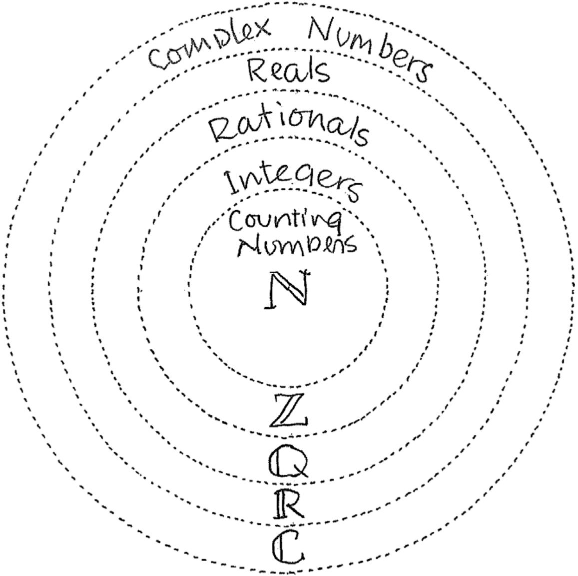

h
1
1


0
1
2

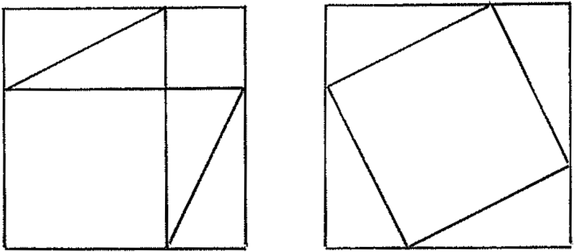
$$
\begin{array}{l}
\qquad\qquad 0.1428571 \\
\dfrac{1}{7} = 7\overline{\big)1.0000000000}\ \ldots \\
\quad \underline{-7} \\
\quad \textcircled{3}0 \\
\quad \underline{-28} \\
\qquad \textcircled{2}0 \\
\qquad \underline{-14} \\
\qquad\ \textcircled{6}0 \\
\qquad\ \underline{-56} \\
\qquad\quad \textcircled{4}0 \\
\qquad\quad \underline{-35} \\
\qquad\qquad \textcircled{5}0 \\
\qquad\qquad \underline{-49} \\
\qquad\qquad\ \textcircled{1}0 \\
\qquad\qquad\quad \underline{-7} \\
\qquad\qquad\quad\ \textcircled{3}
\end{array}
$$

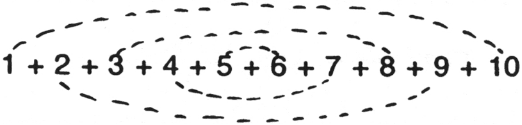


The Complex Plane



The Talisman of the New Pythagoreans

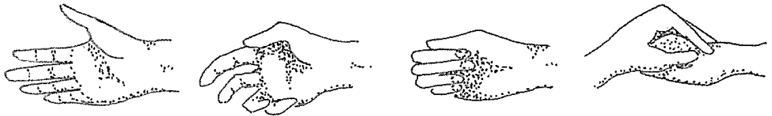


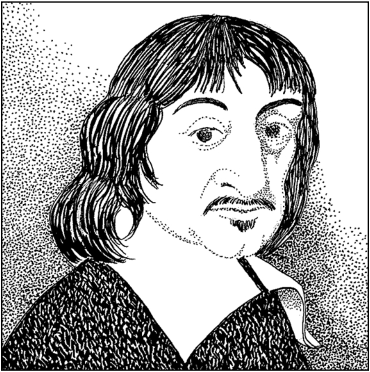


1 + 10 = 11, 2 + 9 = 11, 3 + 8 = 11—in fact, all these pairs will add up to 11! And how many pairs are there? 5—that is, half of 10. So

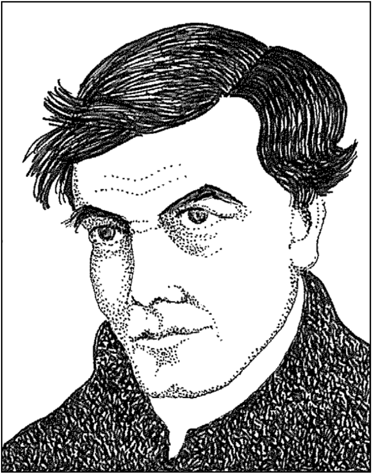
$$1+2+3+4+5+6+7+8+9+10=\left(\frac{10}{2}\right)\cdot(11),$$

$$\text{or } \left(\frac{n}{2}\right)\cdot(n+1).$$





René Descartes (1596–1650), whose interest in mathematics was sparked by a problem he saw posted on a wall in Holland in 1618.

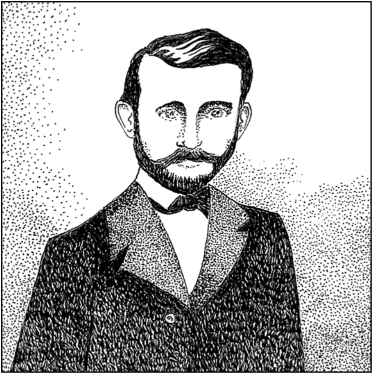

George Peacock (1791–1858)

The Axioms for a Field

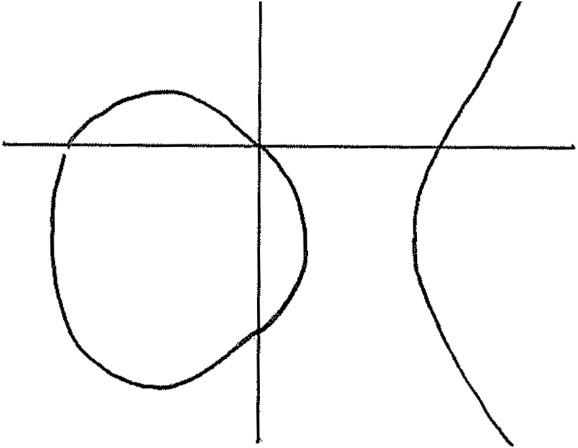
if a, b, and c are numbers, then

Under Addition			Under Multiplication	
A0	$a + b$ is a number	*Closure*	M0	$a \cdot b$ is a number
A1	$a + (b + c) = (a + b) + c$	*Associativity*	M1	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
A2	$a + b = b + a$	*Commutivity*	M2	$a \cdot b = b \cdot a$
A3	there is a number, 0, such that $a + 0 = a$	*Identity*	M3	there is a number, 1, such that $a \cdot 1 = a$; and $1 \neq 0$.
A4	for any number a there is a number, –a, such that $a + (-a) = 0$	*Inverse*	M4	for any number a, except 0, there is a number, $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = 1$

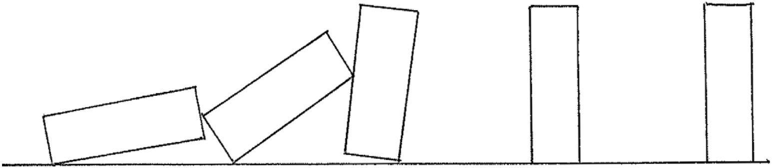
D *Distributivity*: $a \cdot (b + c) = a \cdot b + a \cdot c$.



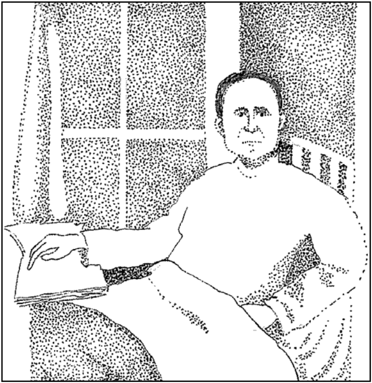
Richard Dedekind
(1831–1916)



There you see all the real solutions to our equation.

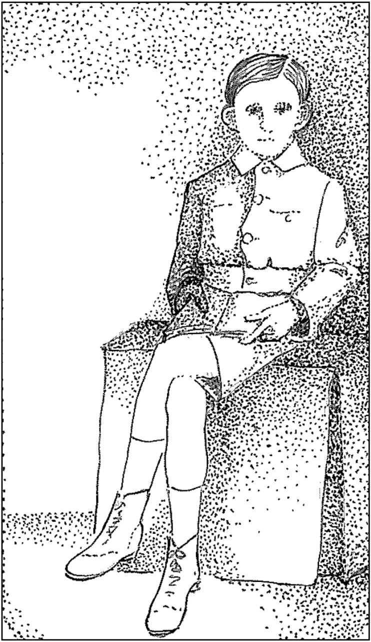


The moon, aged fourteen days and one hour, from a photograph made through a telescope on October 27, 1890. The crater Maurolicus is in the upper-left quadrant, below Tycho. If this orientation bothers you, it isn't that the moon has turned over in the course of a century, but that the telescope lens inverted the image.



L. E. J. Brouwer (1881–1966). Like Descartes and Gauss, he had his best insights in bed.

David Hilbert (1862–1943). From a set of postcards of the Mathematics Department sold by the University of Göttingen to tourists.



Erdos at eight. The book in his hand is most likely not yet The Book.

Before

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46
47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67
68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88
89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106

During

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ 23 ~~24~~ ~~25~~
~~26~~ ~~27~~ ~~28~~ 29 ~~30~~ 31 ~~32~~ ~~33~~ ~~34~~ ~~35~~ ~~36~~ 37 ~~38~~ ~~39~~ ~~40~~ 41 ~~42~~ 43 ~~44~~ ~~45~~ ~~46~~
47 ~~48~~ ~~49~~ ~~50~~ ~~51~~ ~~52~~ 53 ~~54~~ ~~55~~ ~~56~~ ~~57~~ ~~58~~ 59 ~~60~~ 61 ~~62~~ ~~63~~ ~~64~~ ~~65~~ ~~66~~ 67
~~68~~ ~~69~~ ~~70~~ 71 ~~72~~ 73 ~~74~~ ~~75~~ ~~76~~ ~~77~~ ~~78~~ 79 ~~80~~ ~~81~~ ~~82~~ 83 ~~84~~ ~~85~~ ~~86~~ ~~87~~ ~~88~~
89 ~~90~~ ~~91~~ ~~92~~ ~~93~~ ~~94~~ ~~95~~ ~~96~~ 97 ~~98~~ ~~99~~ ~~100~~ 101 ~~102~~ 103 ~~104~~ ~~105~~ ~~106~~

After

2 3 5 7 11 13 17 19 23

29 31 37 41 43

47 53 59 61 67

71 73 79 83

89 97 101 103

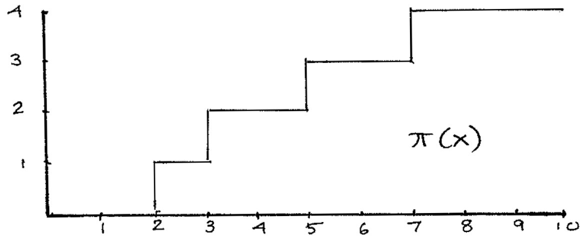
Between	The Number of Primes Is
1 and 100	25
100 and 200	21
200 and 300	16
300 and 400	16
400 and 500	17
500 and 600	14
600 and 700	16
700 and 800	14
800 and 900	15
900 and 1000	14

Between	The Number of Primes Is
1,000,000 and 1,000,100	6
1,000,100 and 1,000,200	10
1,000,200 and 1,000,300	8
1,000,300 and 1,000,400	8
1,000,400 and 1,000,500	7
1,000,500 and 1,000,600	7
1,000,600 and 1,000,700	10
1,000,700 and 1,000,800	5
1,000,800 and 1,000,900	6
1,000,900 and 1,001,000	8

Between	The Number of Primes Is
10,000,000 and 10,000,100	2
10,000,100 and 10,000,200	6
10,000,200 and 10,000,300	6
10,000,300 and 10,000,400	6
10,000,400 and 10,000,500	5
10,000,500 and 10,000,600	4
10,000,600 and 10,000,700	7
10,000,700 and 10,000,800	10
10,000,800 and 10,000,900	9
10,000,900 and 10,001,000	6

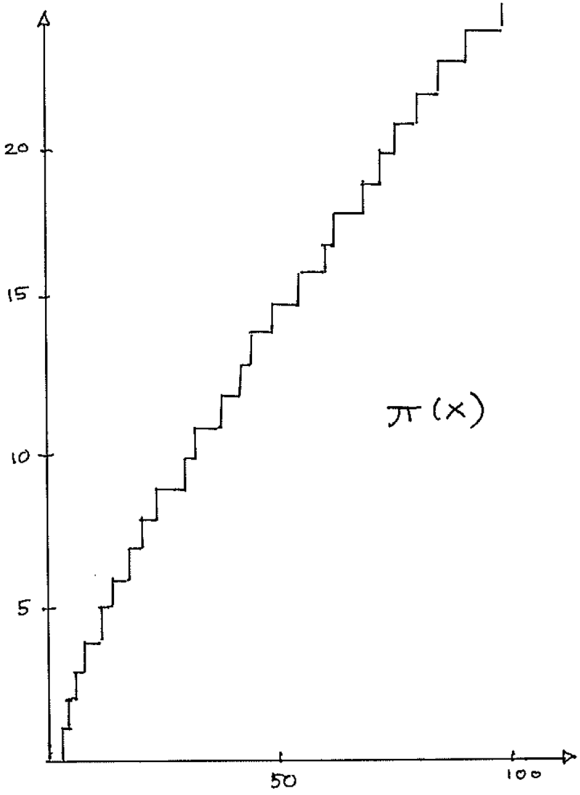
Between	The Number of Primes Is
10^{12} and $10^{12} + 100$	4
$10^{12} + 100$ and $10^{12} + 200$	6
$10^{12} + 200$ and $10^{12} + 300$	2
$10^{12} + 300$ and $10^{12} + 400$	4
$10^{12} + 400$ and $10^{12} + 500$	2
$10^{12} + 500$ and $10^{12} + 600$	4
$10^{12} + 600$ and $10^{12} + 700$	3
$10^{12} + 700$ and $10^{12} + 800$	5
$10^{12} + 800$ and $10^{12} + 900$	1
$10^{12} + 900$ and $10^{12} + 1000$	6

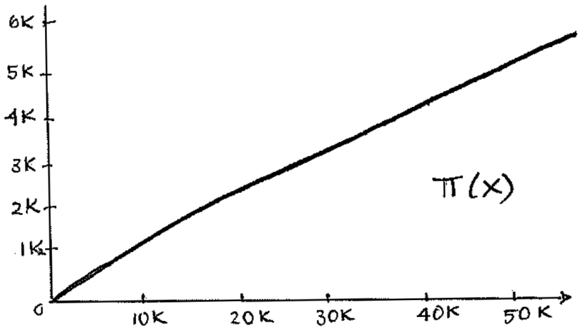
Carl Friedrich Gauss (1777–1855), a mason's son and the master builder of mathematics.

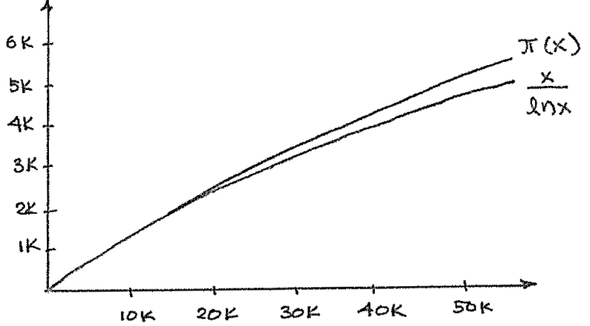


Only the horizontal "treads" matter in this step-graph. The vertical "risers" are conventionally put in just to give it a coherent shape.

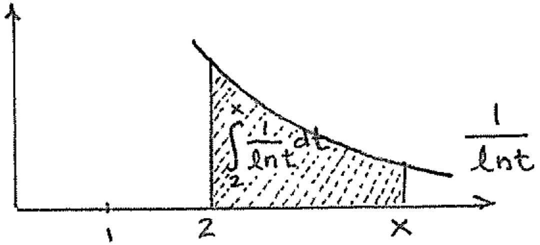
Here is the graph of $\pi(x)$ for x up to 100 (in order to accommodate the slow growth of the primes, we have shrunk the units on the vertical axis until those on the horizontal axis are about seven times their size, so that the graph looks much steeper than it should):

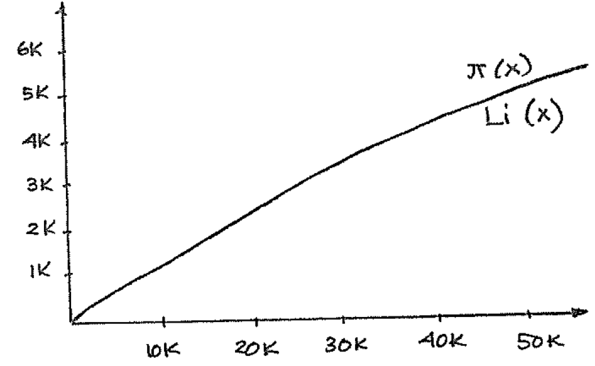


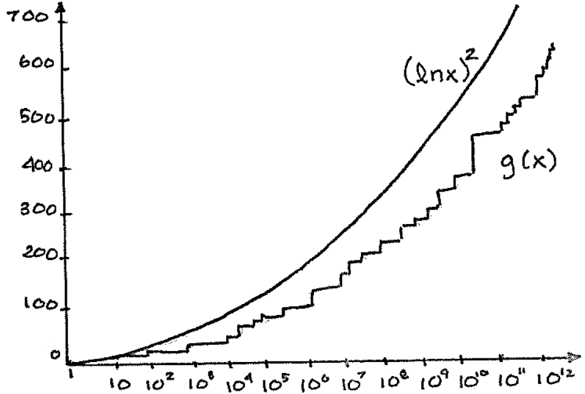

6K
5K
4K
3K
2K
1K
0
10K
20K
30K
40K
50K
π(x)



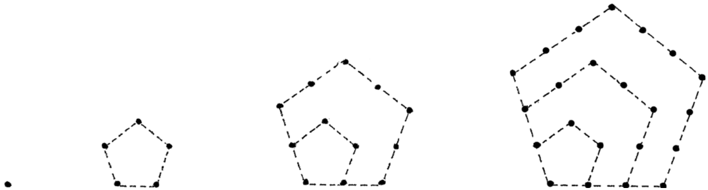
6K
5K
4K
3K
2K
1K
10K
20K
30K
40K
50K
π(x)
x/ln x


$\int_2^x \frac{1}{\ln t} dt$
$\frac{1}{\ln t}$
1
2
x

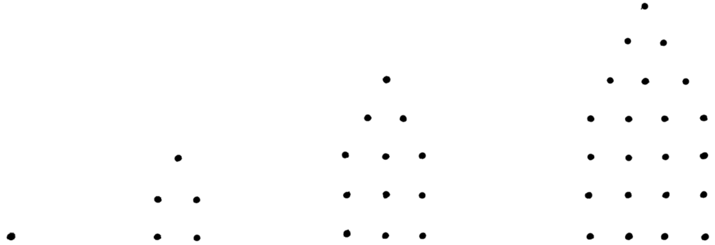

6K
5K
4K
3K
2K
1K
10K
20K
30K
40K
50K
π (x)
Li (x)



700
600
500
400
300
200
100
0
1
10
10²
10³
10⁴
10⁵
10⁶
10⁷
10⁸
10⁹
10¹⁰
10¹¹
10¹²
(ln x)²
g(x)



1 5 12 22



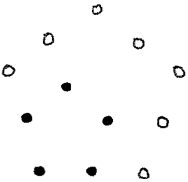
1 $\quad$ $1 + 4 = 5$ $\quad$ $3 + 9 = 12$ $\quad$ $6 + 16 = 22$

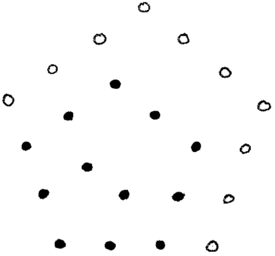
That is:

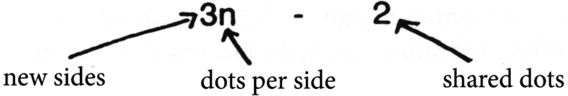
△1 + □2 = ⬠2 $\qquad$ △2 + □3 = ⬠3 $\qquad$ △3 + □4 = ⬠4

and indeed

△4 + □5 $= 10 + 25 = 35$, which is ⬠5, and so, it seems, on.






3n - 2
new sides
dots per side
shared dots

Old Name	New Name	First	Second	Third	Fourth	Fifth
triangular	3-gonal	1	3	6	10	15
square	4-gonal	1	4	9	16	25
pentagonal	5-gonal	1	5	12	22	35
hexagonal	6-gonal	1	6	15	28	45
heptagonal	7-gonal	1	7	18	34	55

k-gonal	First	Second	Third	Fourth	Fifth
3-gonal	1	1+2	1+2+3	1+2+3+4	1+2+3+4+5
4-gonal	1	1+3	1+3+5	1+3+5+7	1+3+5+7+9
5-gonal	1	1+4	1+4+7	1+4+7+10	1+4+7+10+13
6-gonal	1	1+5	1+5+9	1+5+9+13	1+5+9+13+17
7-gonal	1	1+6	1+6+11	1+6+11+16	1+6+11+16+ 21

$$1+\frac{1}{2}=\frac{3}{2}$$

$$1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{31}{16}$$

$$1+\frac{1}{3}=1\frac{1}{3}$$

$$1+\frac{1}{3}+\frac{1}{9}=1\frac{4}{9}$$

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}=1\frac{13}{27}$$

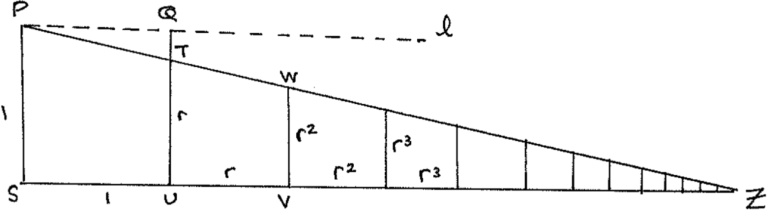
$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}=1\frac{40}{81}$$

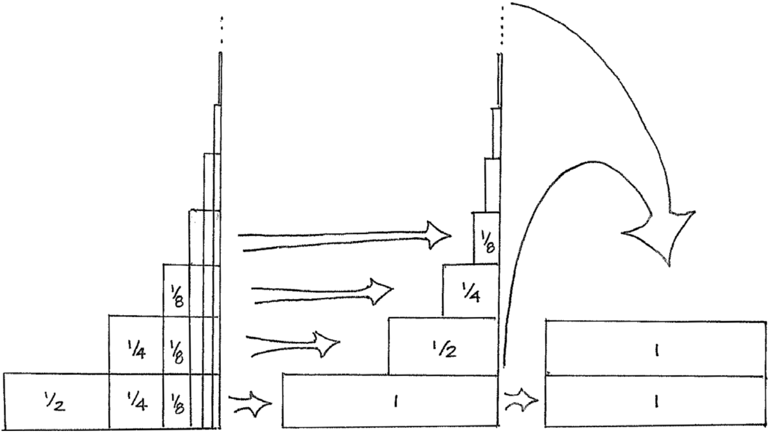
$$1+\frac{1}{4}+\frac{5}{4}=1\frac{1}{4}$$

$$1+\frac{1}{4}+\frac{1}{16}=1\frac{5}{16}$$

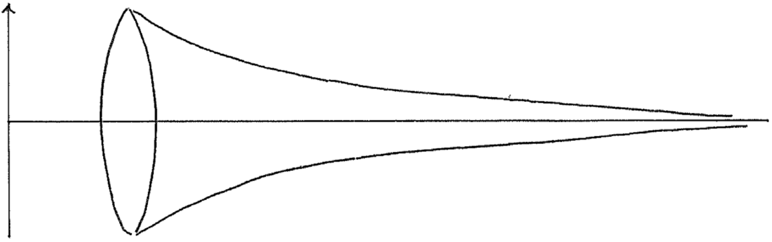
$$1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}=1\frac{21}{64}$$

$$1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}=1\frac{85}{256}$$


P
Q
l
T
W
1
r
r^2
r^3
S
1
U
r
V
r^2
r^3
Z

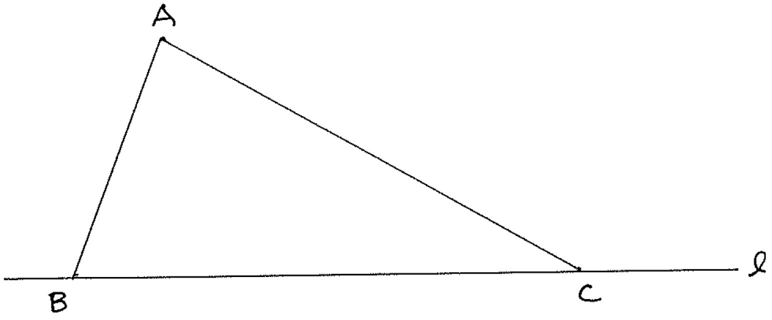


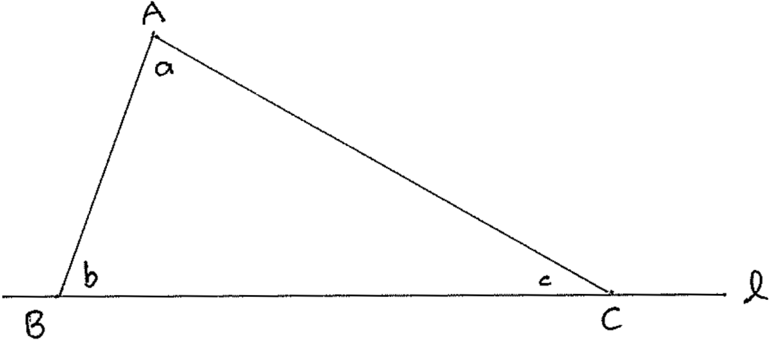
1/2
1/4
1/4
1/8
1/8
1/8
1
1/2
1/4
1/8
1
1

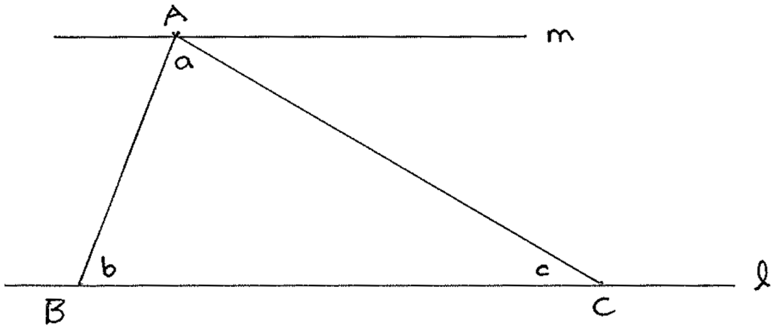


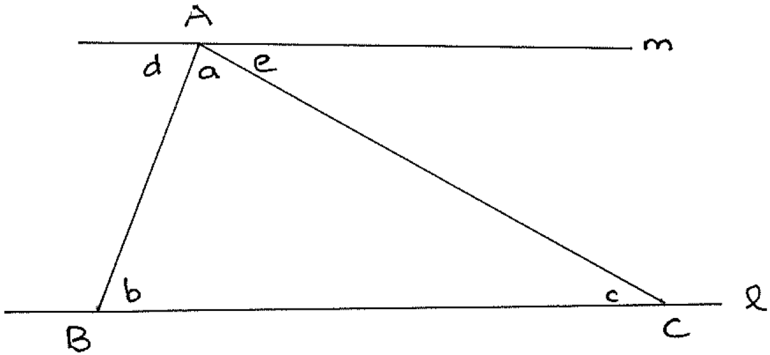
$$\underset{1}{1}+\underset{\frac{1}{2}}{\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\substack{>\frac{1}{2} \\ \text{compare} \\ \text{each term} \\ \text{with } \frac{1}{4}}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\substack{>\frac{1}{2} \\ \text{compare} \\ \text{each term} \\ \text{with } \frac{1}{8}}}+\ldots$$

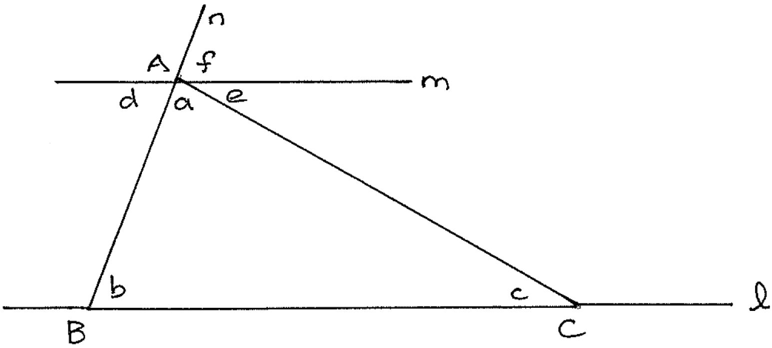
Hermann Minkowski
(1864–1909)

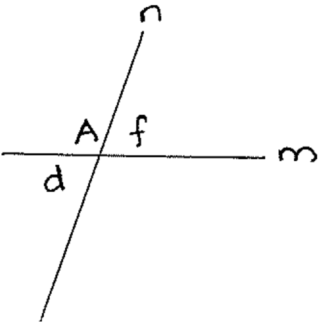

A
l
B
C

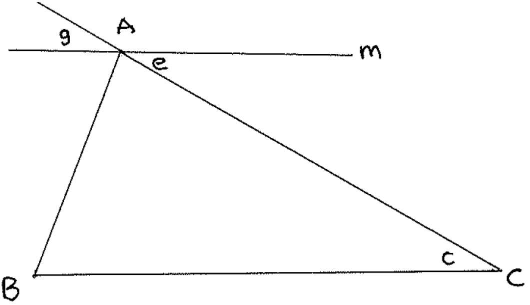

A
a
b
c
B
C
l


A
m
a
b
c
l
B
C

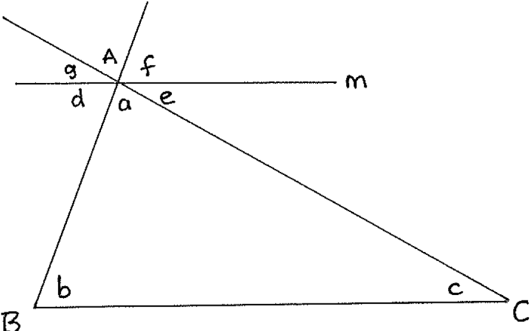

A
m
d
a
e
b
c
ℓ
B
C


n
A
f
m
d
a
e
b
c
l
B
C

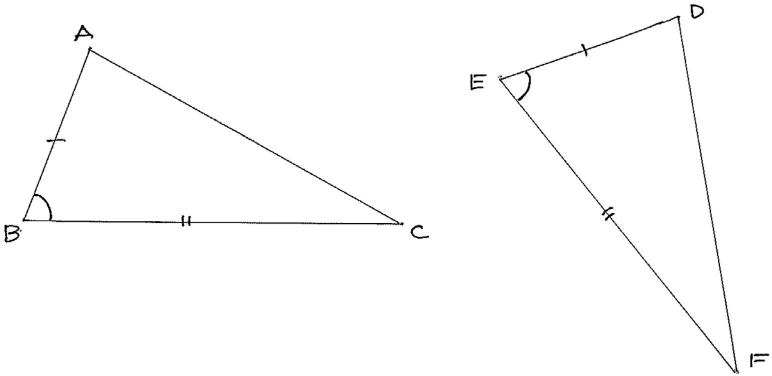

n
A
f
m
d

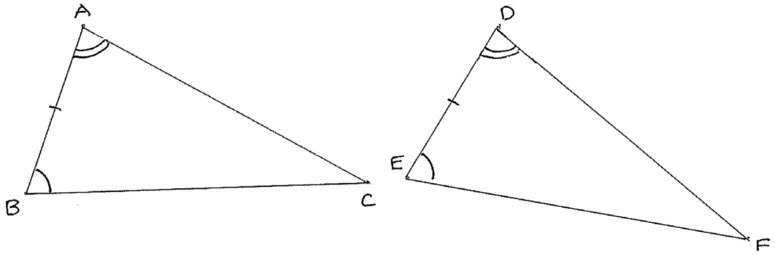


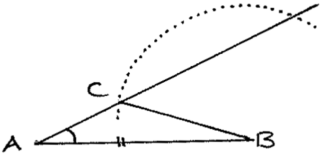
A
g
m
e
c
B
C

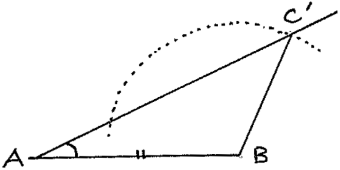


A
g
f
m
d
a
e
b
c
B
C

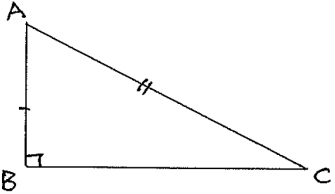

A
B
C
D
E
F

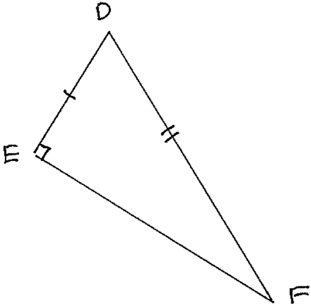

A
B
C
D
E
F



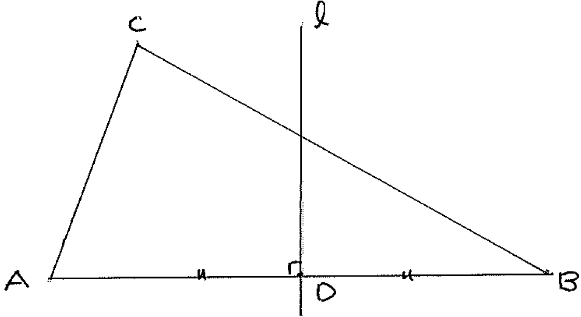


In the special case of right triangles, the equality of one pair of legs, and of the respective hypotenuses, suffices:

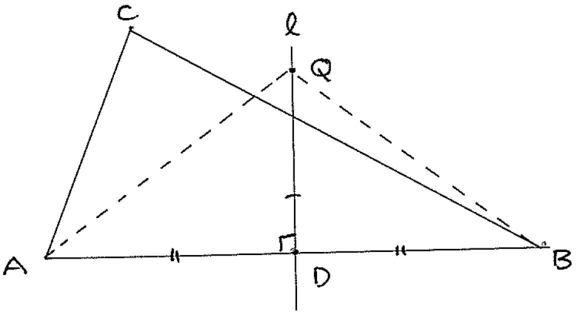


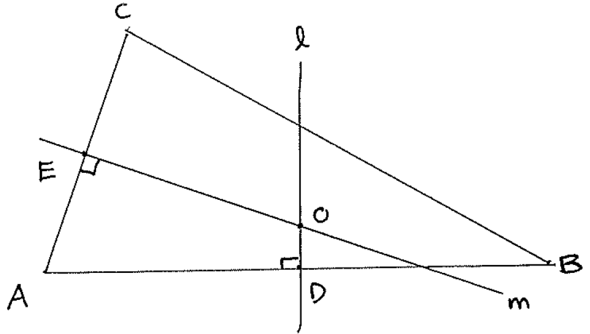


(AB = DE, AC = DF, so right ΔABC ≅ right ΔDEF)

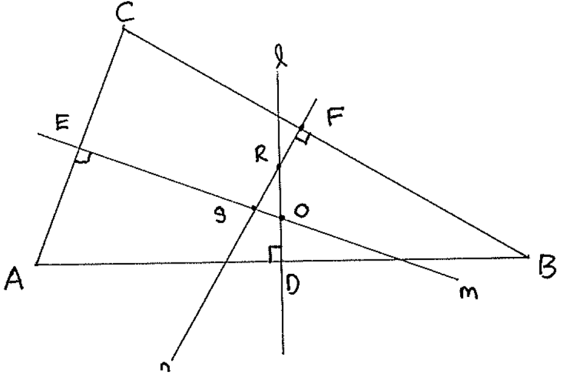


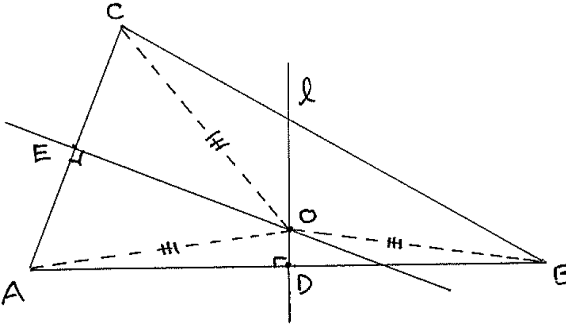
C
l
A
D
B

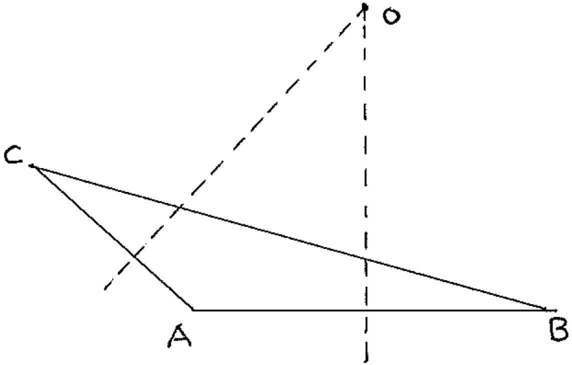

C
l
Q
A
B
D

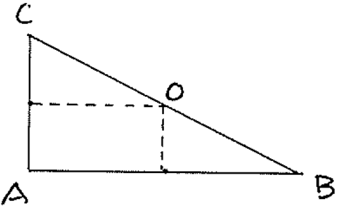


C
l
E
O
B
A
D
m

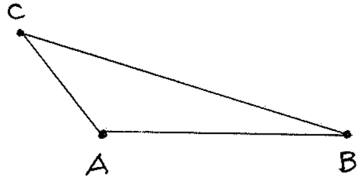

C
l
F
E
R
g
O
A
D
B
m
n

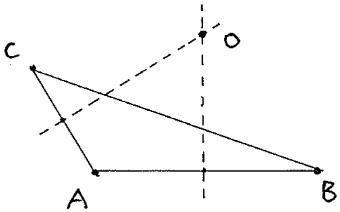

C
l
E
O
A
D
B

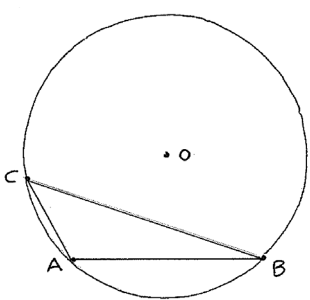

O
C
A
B

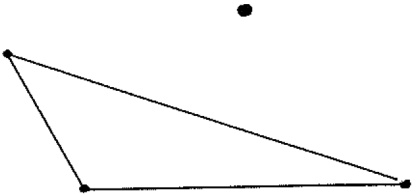

C
O
A
B

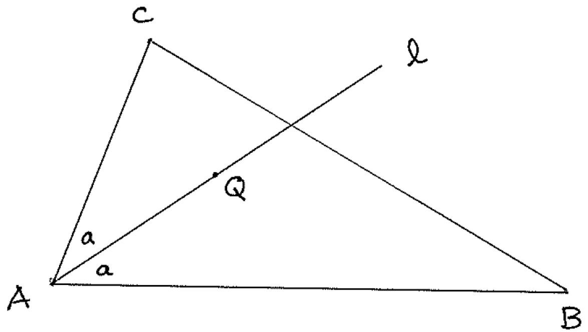

C
A
B

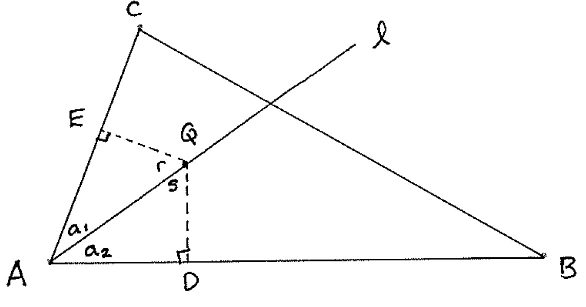

C
A
B


C
O
A
B

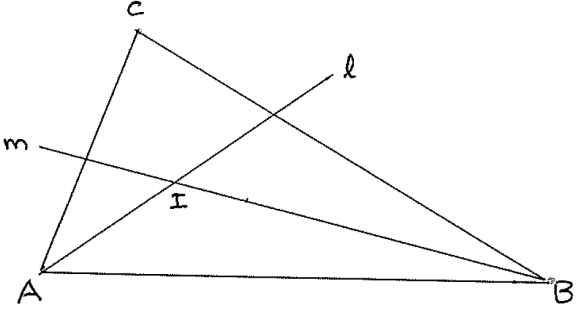

O
C
A
B

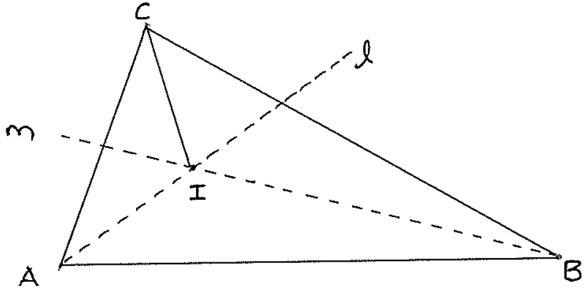



C
l
Q
a
a
A
B

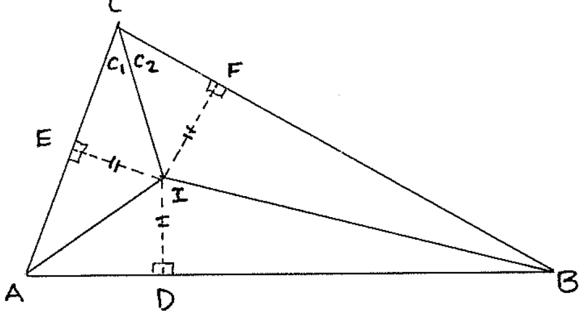


C
l
E
Q
r
s
a1
a2
A
D
B

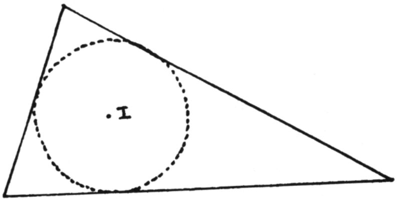

C
ℓ
m
I
A
B

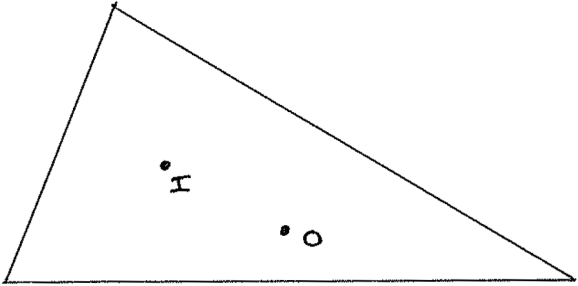


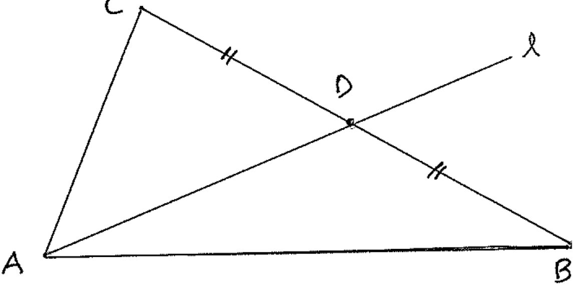
C
ℓ
3
I
A
B

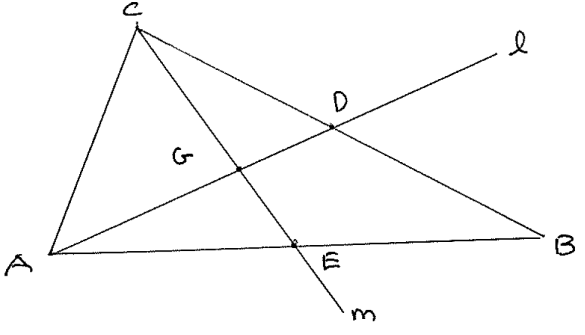


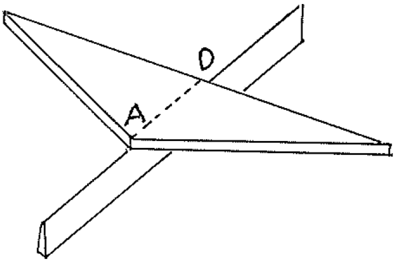
C
c1
c2
F
E
I
A
D
B

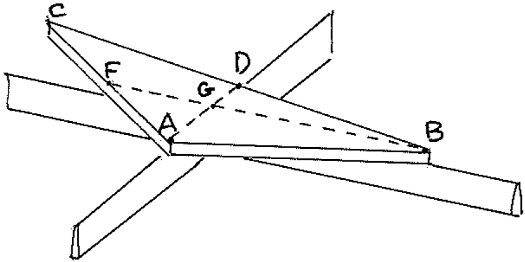

I


H
O

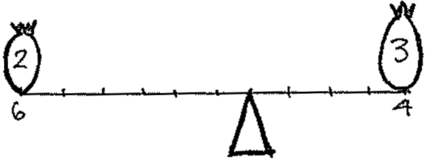

C
l
D
A
B


C
l
D
G
A
E
B
m

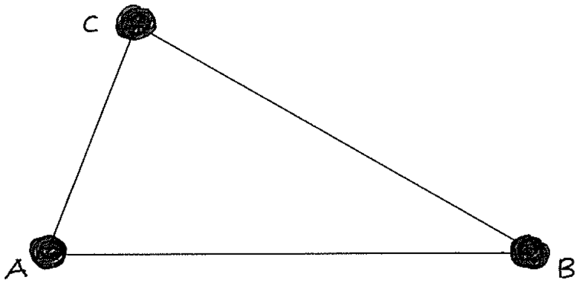

D
A

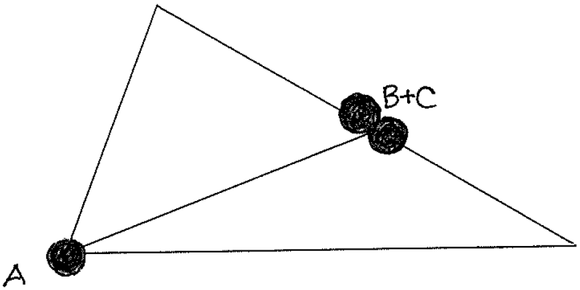


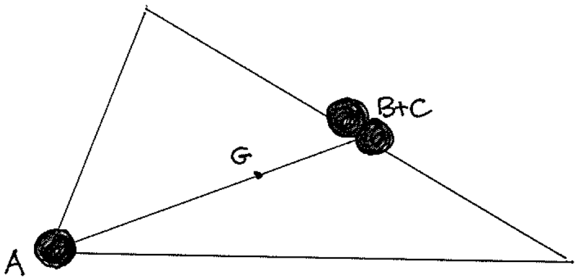
C
F
G
D
A
B



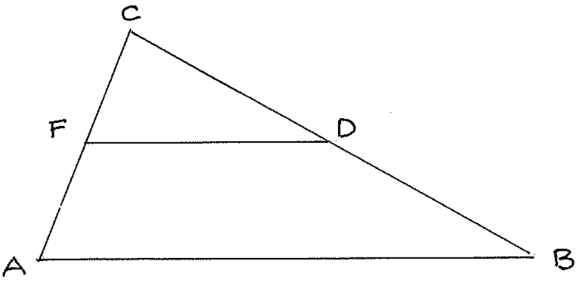
$$2 \times 6 = 3 \times 4$$


C
A
B

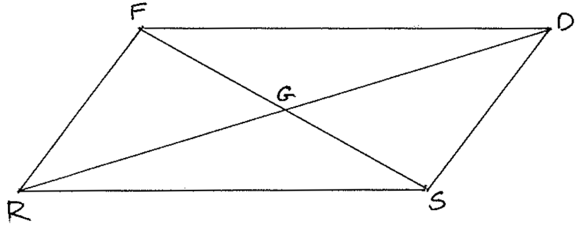

B+C
A



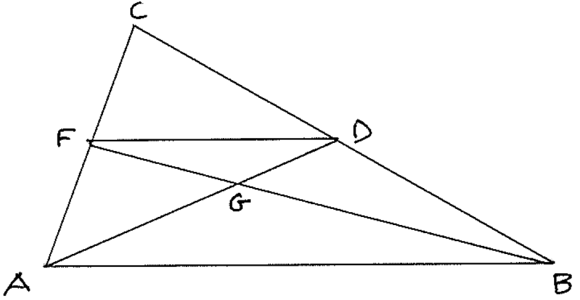
B+C
G
A

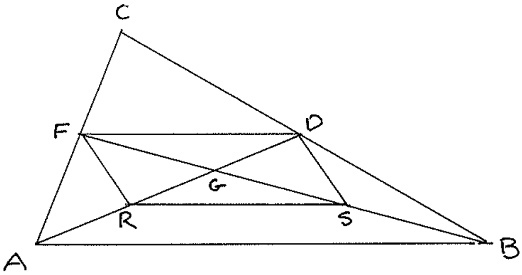


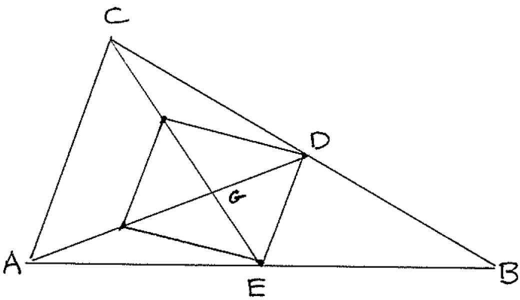
(here the line FD is parallel to AB, and half its length)

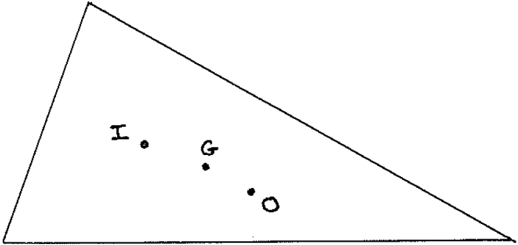


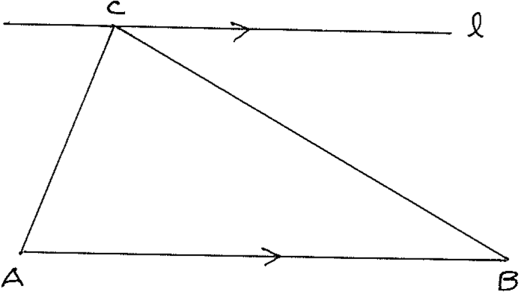
$(RG = GD, FG = GS)$


C
F
D
G
A
B

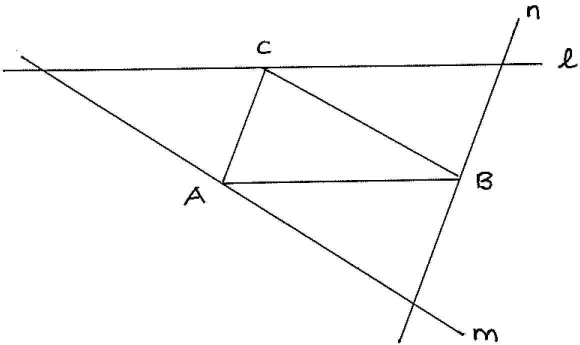

C
F
D
G
R
S
A
B


C
D
G
A
E
B

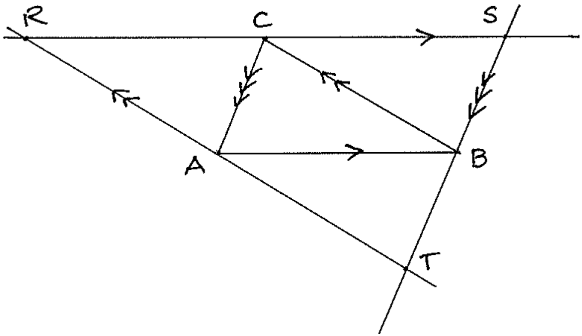

I
G
O

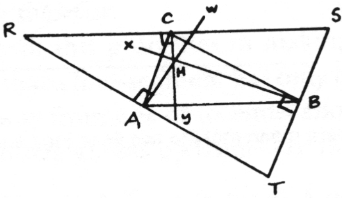


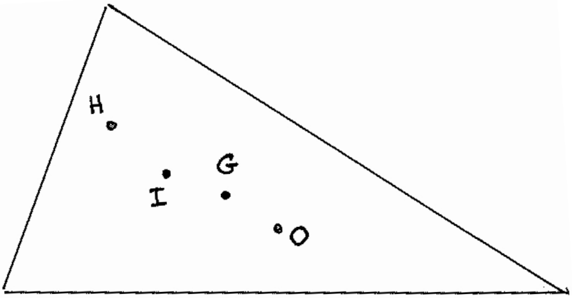
(our little arrows here mean that the two lines they are on are parallel).

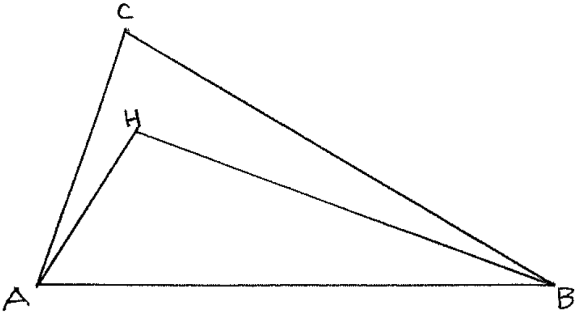


These new lines form a new triangle; we'll label its vertices R, S, and T.

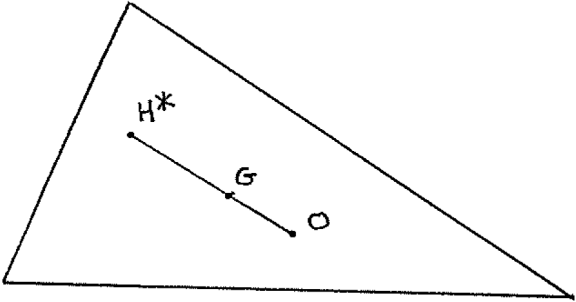


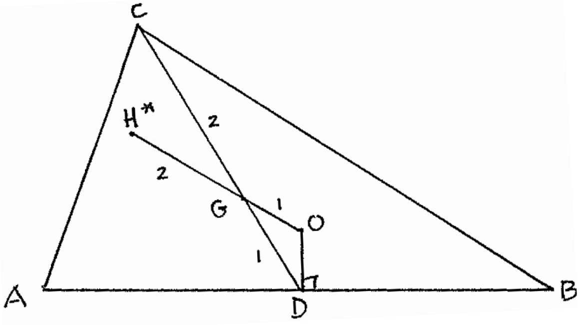

C
W
S
R
X
H
B
A
y
T

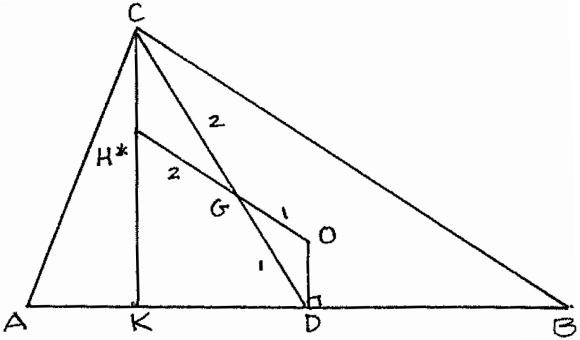

H
G
I
O

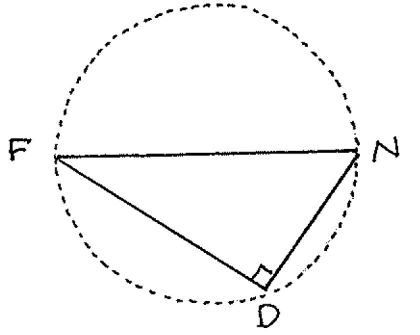

C
H
A
B

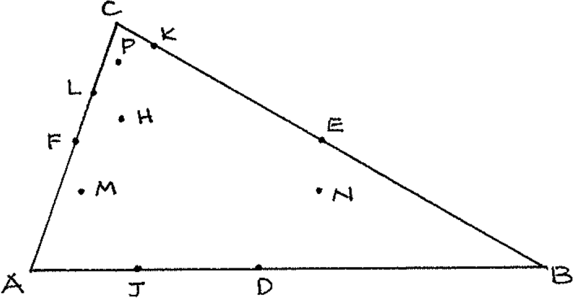
Leonhard Euler (1707–1783), father of thirteen and endlessly productive in mathematics.


H*
G
O

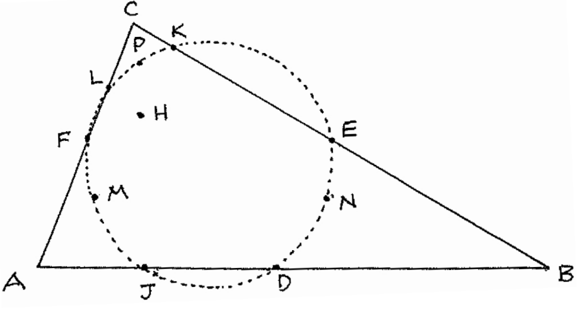

C
H*
2
2
G
1
O
1
A
D
B

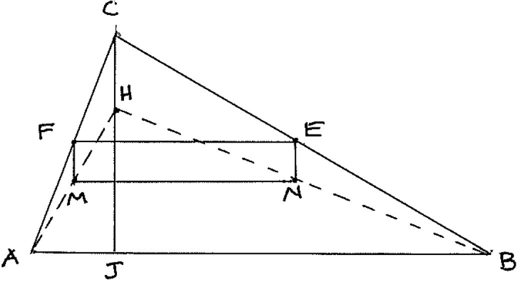

C
H*
2
2
G
1
O
1
A
K
D
B

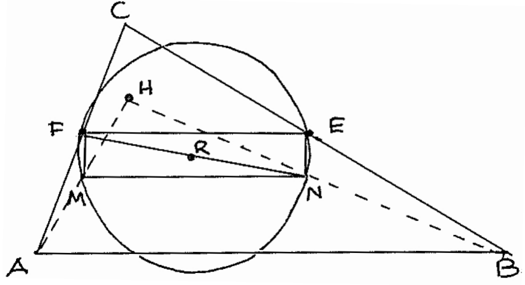

F
N
D



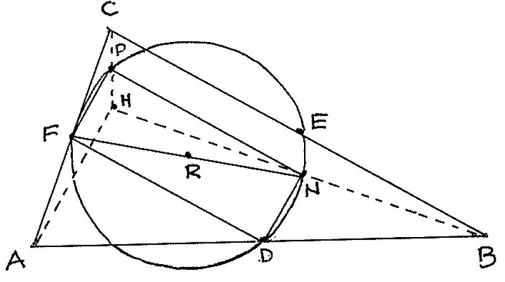
C
K
P
L
H
E
F
M
N
A
J
D
B

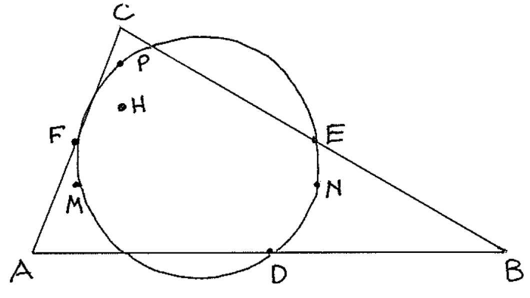

C
K
P
L
H
E
F
M
N
A
J
D
B

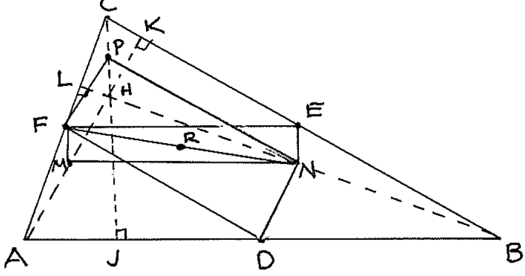

C
H
F
E
M
N
A
J
B

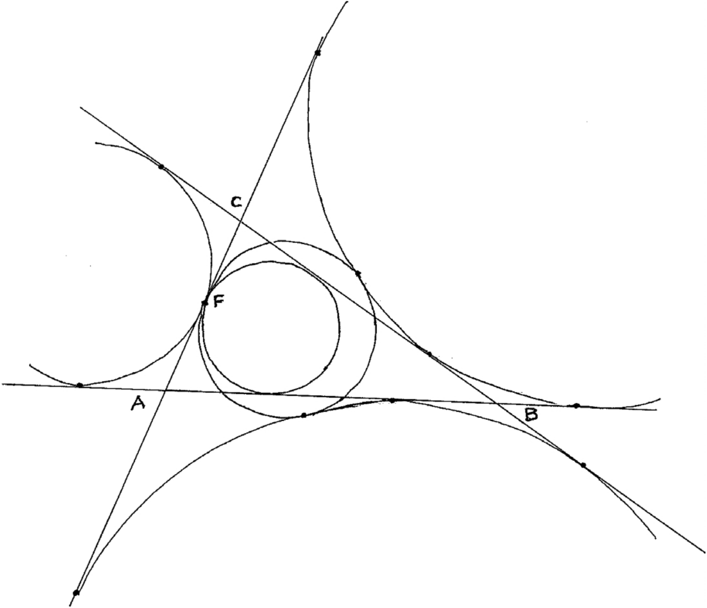


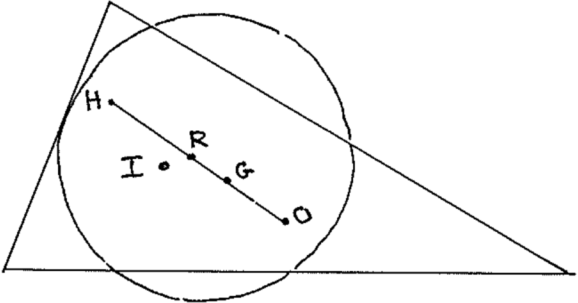
C
H
F
E
R
M
N
A
B

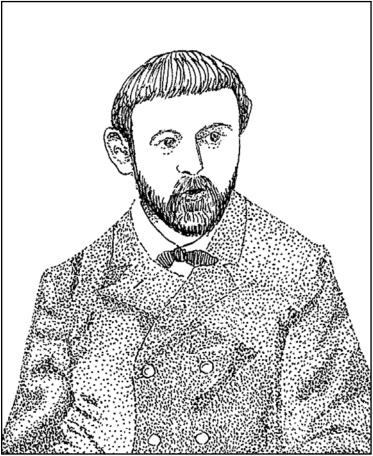

C
P
H
E
F
R
N
A
D
B


C
P
H
F
E
M
N
A
D
B

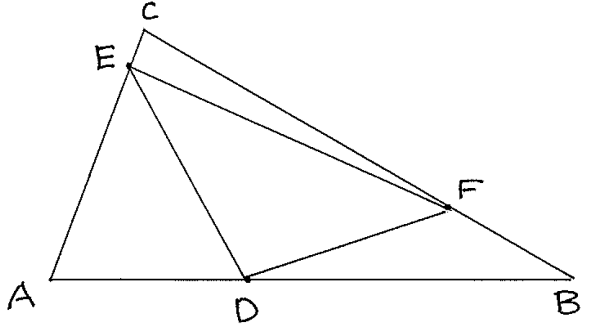

C
K
P
L
H
E
F
R
M
N
A
J
D
B

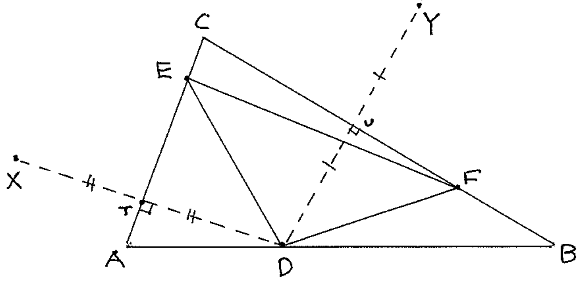

C
F
A
B


H
R
I
G
O

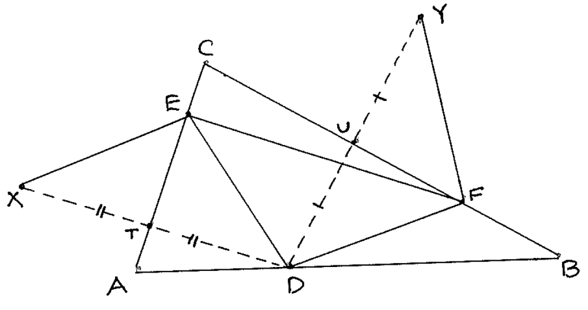


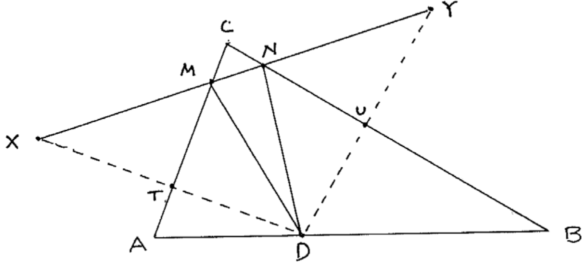
Henri Poincaré (1854–1912)


C
E
F
A
D
B

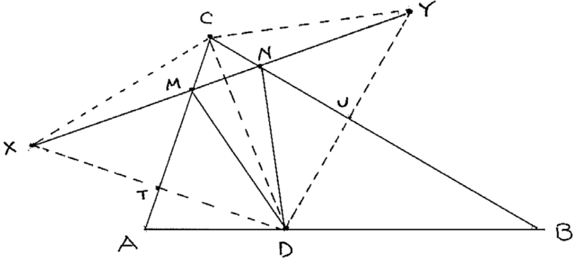


C
Y
E
X
F
A
D
B

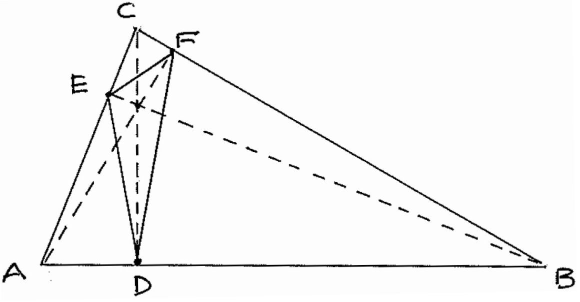

Y
C
E
U
X
F
T
A
D
B



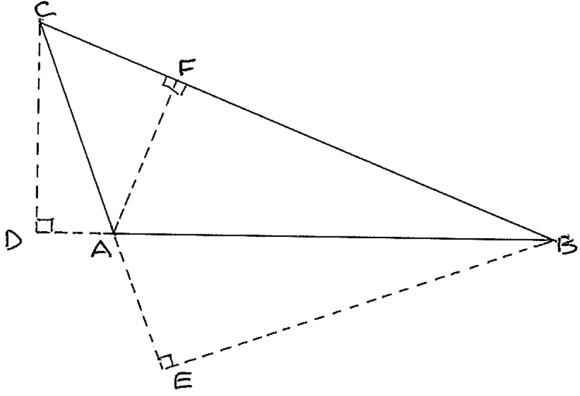
C
N
Y
M
U
X
T
A
D
B

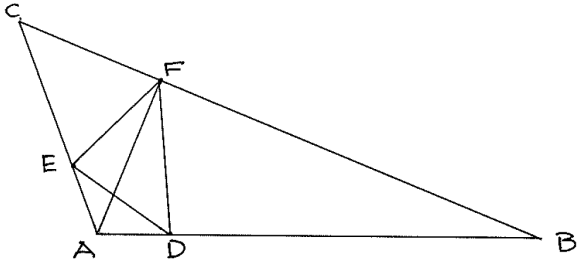


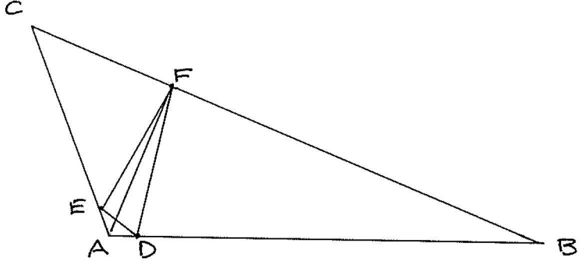
C
Y
N
M
U
X
T
A
D
B

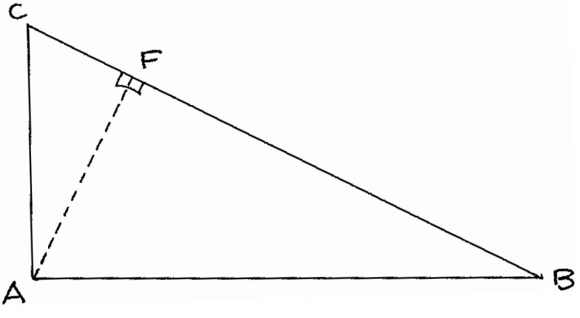


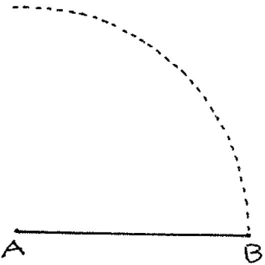
C
F
E
A
D
B

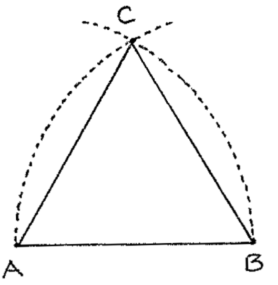

C
F
D
A
B
E

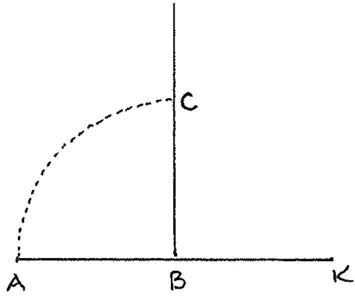

C
F
E
A
D
B

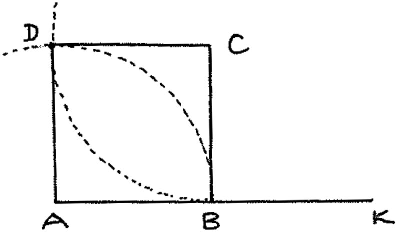

C
F
E
A
D
B

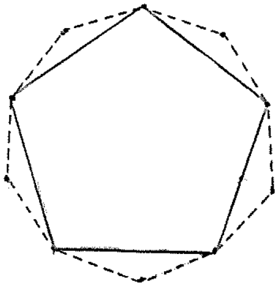

C
F
A
B

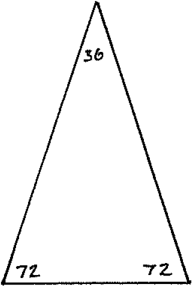

A
B

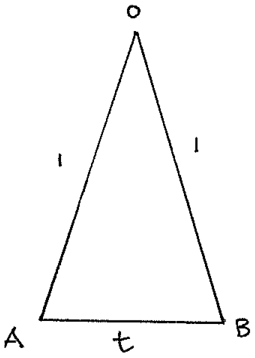

C
A
B

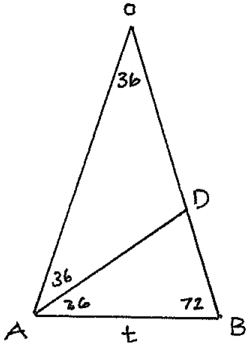

C
A
B
K


D
C
A
B
K

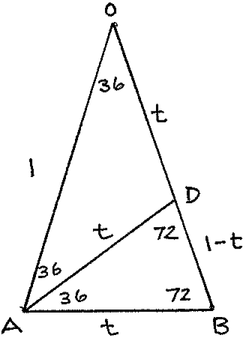


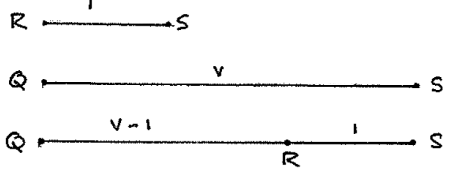

36
72
72

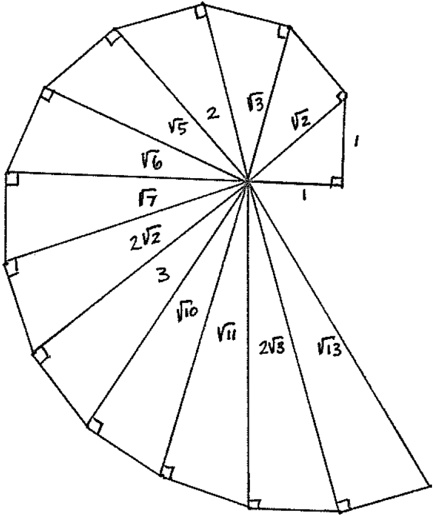

O
1
1
A
t
B



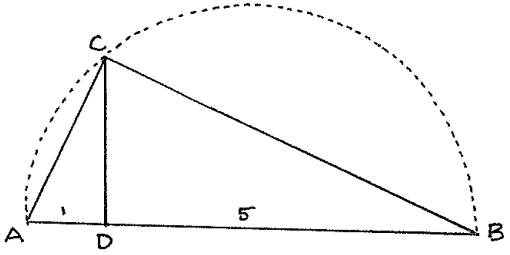
o
36
D
36
36
72
A
t
B

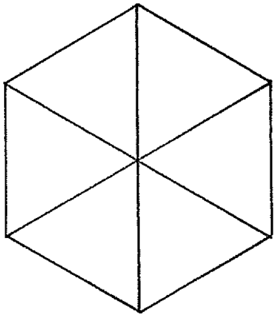

O
36
t
1
D
t
72
1-t
36
36
72
A
t
B


1
R
S
Q
V
S
Q
V - 1
R
1
S

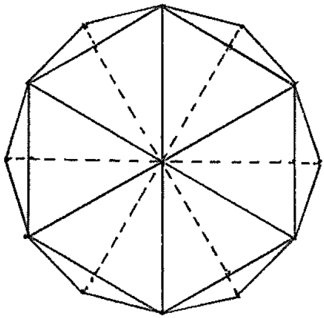


1
1
√2
√3
2
√5
√6
√7
2√2
3
√10
√11
2√3
√13

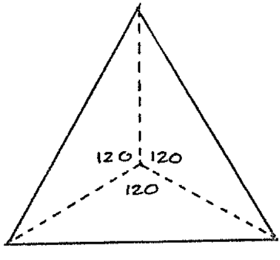

C
A
1
D
5
B



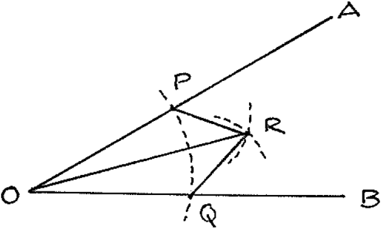
Suddenly it becomes clear that we can construct a 12-sided polygon (dodecagon) if we can bisect the central angles of the hexagon—

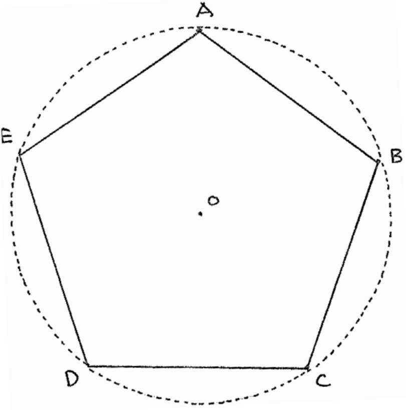


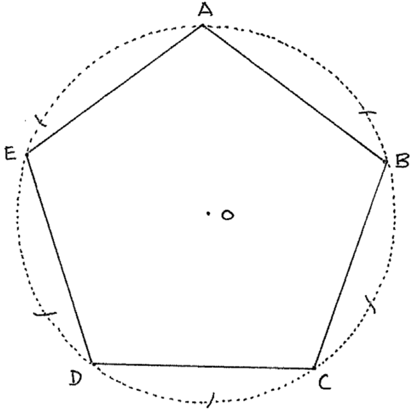
and indeed that we could have found the hexagon by bisecting the central angles of the triangle:



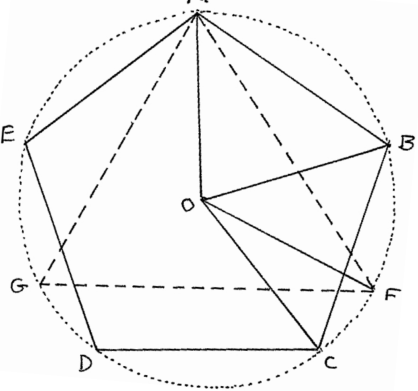
so if we could bisect an angle, every constructed n-gon would give us a 2n-gon for free—and angle-bisection falls readily to compass and straightedge. To bisect ∠AOB, swing any arc with center O, meeting AO at P and BO at Q.

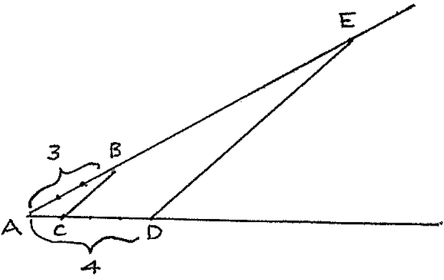

A
P
R
O
B
Q

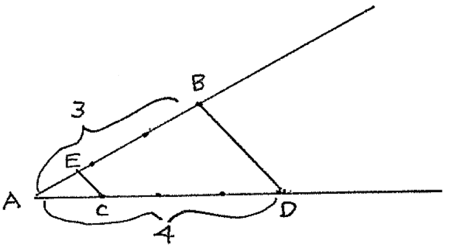

A
B
C
D
E
O



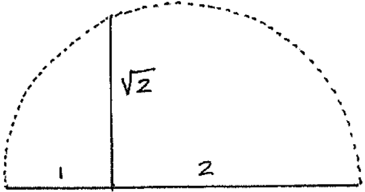
A
B
C
D
E
O


A
B
C
D
E
F
G
O

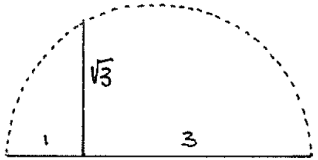

E
3
B
A
C
D
4



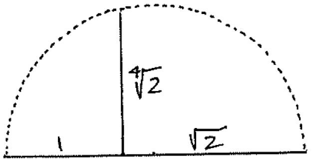
B
3
E
A
C
D
4



√2
1
2

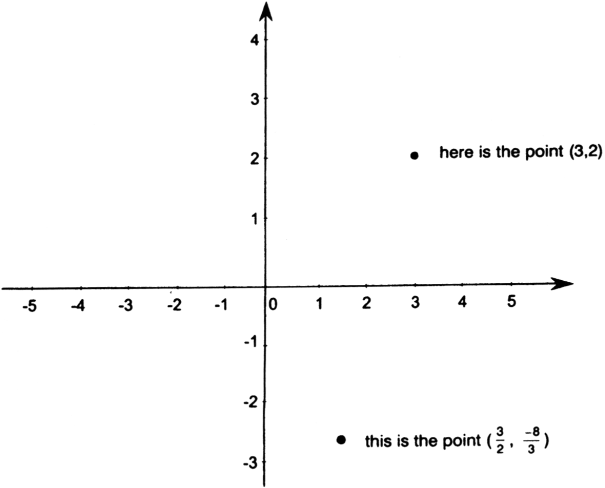


√3
1
3

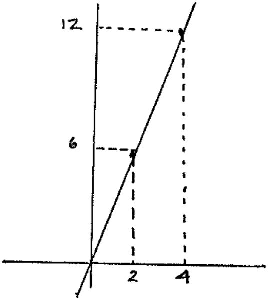


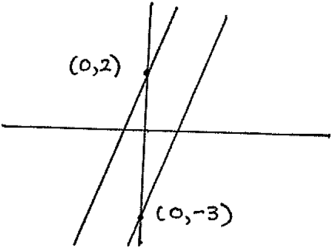
∜2
1
√2

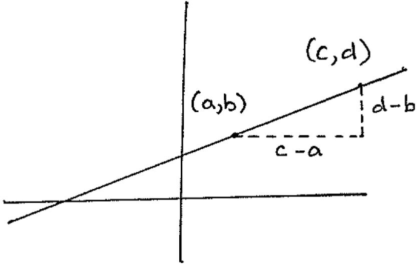
$$
\begin{array}{ccccccc}
F & \rightarrow & F_1 = F[\sqrt{2}] & \rightarrow & F_2 = F_1[\sqrt{3}] & \rightarrow & F_3 = F_2[\sqrt{5}] \quad \rightarrow \ldots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & F_{1,1} = F_1[\sqrt[4]{2}] & & F_{2,1} = F_2[\sqrt[4]{3}] & & F_{3,1} = F_3[\sqrt[4]{5}] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & F_{1,2} = F_{1,1}[\sqrt[8]{2}] & & F_{2,2} = F_{2,1}[\sqrt[8]{3}] & & F_{3,2} = F_{3,1}[\sqrt[8]{5}] \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & F_{1,3} = F_{1,2}[\sqrt[16]{2}] & & F_{2,3} = F_{2,2}[\sqrt[16]{3}] & & F_{3,3} = F_{3,2}[\sqrt[16]{5}]
\end{array}
$$

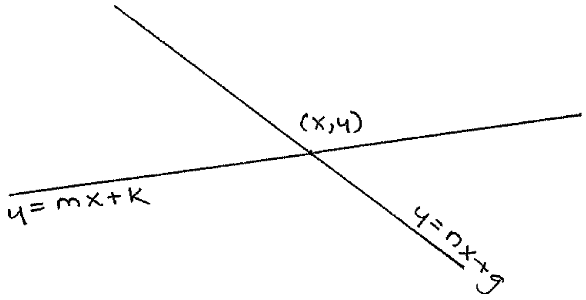


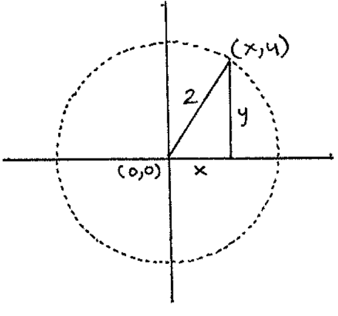
here is the point (3,2)
this is the point $(\frac{3}{2}, \frac{-8}{3})$
4
3
2
1
0
-1
-2
-3
-5 -4 -3 -2 -1 0 1 2 3 4 5

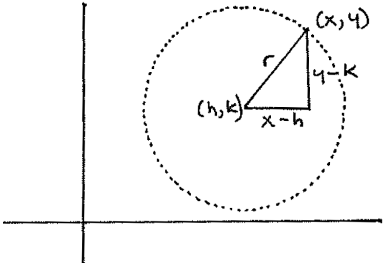

12
6
2
4


(0,2)
(0,-3)

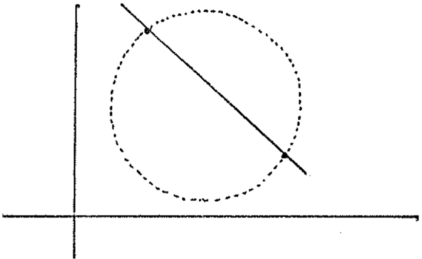

(c, d)
(a,b)
d-b
c -a

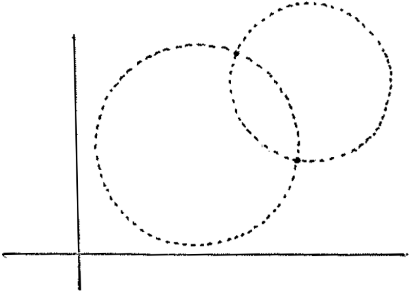

(x,y)
y= mx+K
y=nx+g

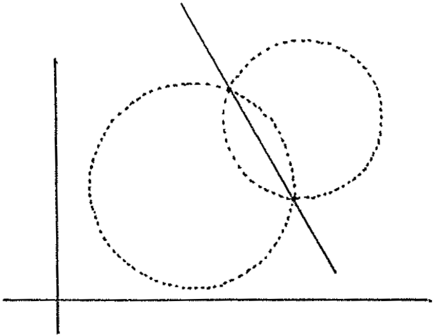

(x,y)
2
y
(0,0)
x

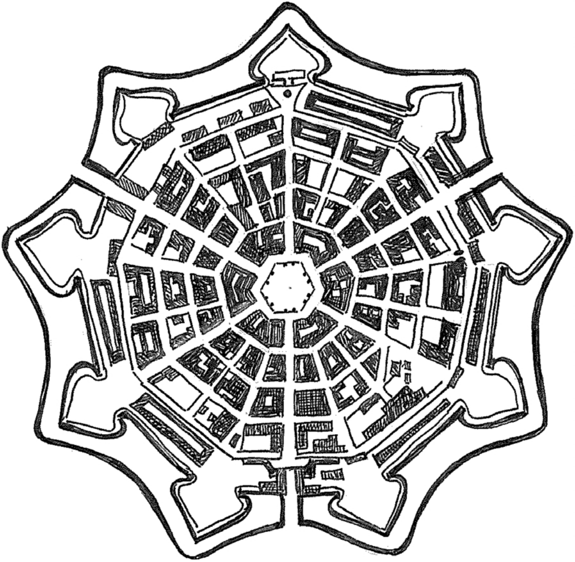


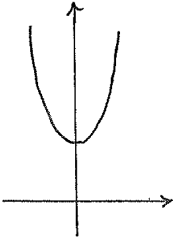
(x, y)
r
y - k
(h, k)
x - h



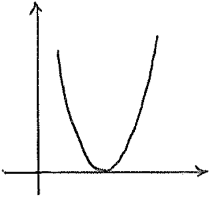




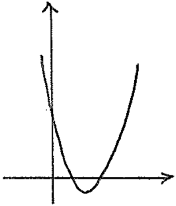




$f(x) = x^2 + 3$

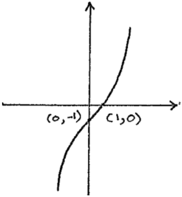


$f(x) = x^2 - 8x + 16$

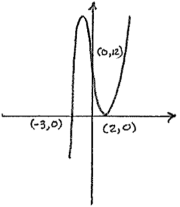


$f(x) = x^2 - 5x + 6$

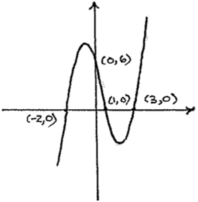
Cubic functions can have one, two, or three roots, but the shape of their graphs forbids their having none.



$f(x) = x^3 - x^2 + x - 1$

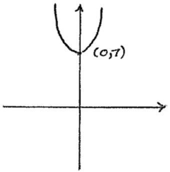


$f(x) = x^3 - x^2 - 8x + 12$

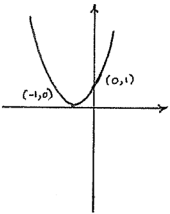


$f(x) = x^3 - 2x^2 - 5x + 6$

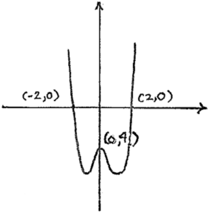
Quartic functions can have no, one, two, three, or four roots,



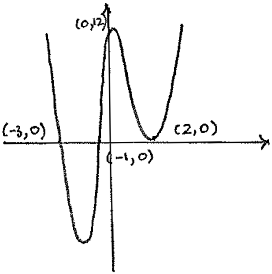
$f(x) = x^4 + 7$



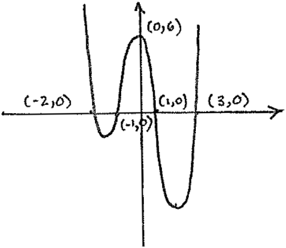
$f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$



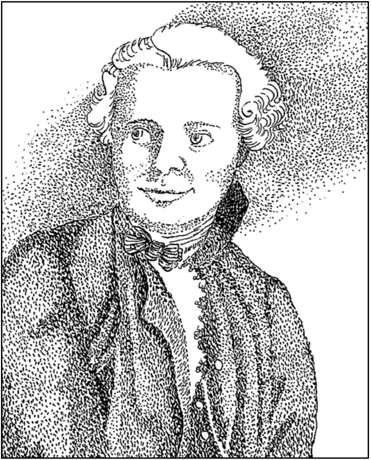
$f(x) = x^4 - 3x^2 - 4$



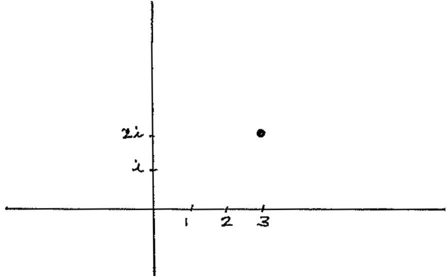
$f(x) = x^4 - 9x^2 + 4x + 12$



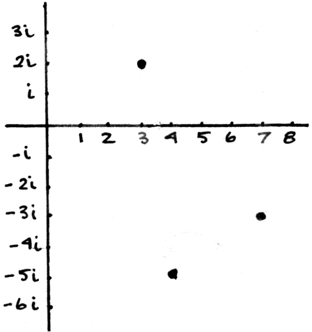
$f(x) = x^4 - x^3 - 7x^2 + x + 6$

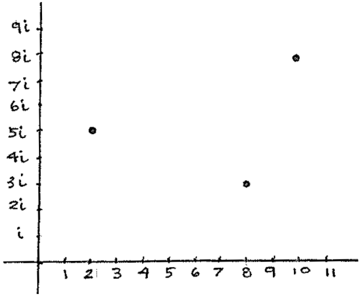


D'Alembert (1717–1783)

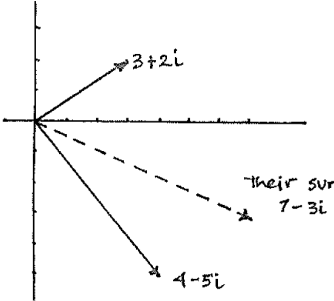


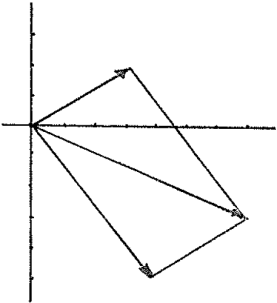
2i
i
1
2
3

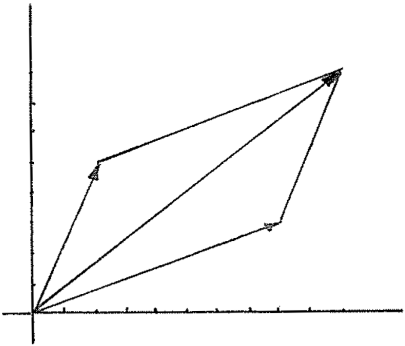

3i
2i
i
1
2
3
4
5
6
7
8
-i
-2i
-3i
-4i
-5i
-6i

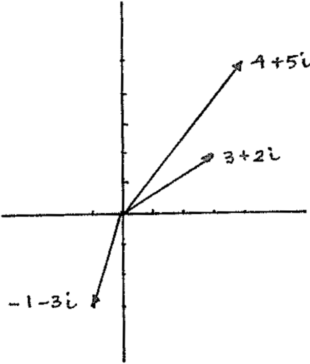


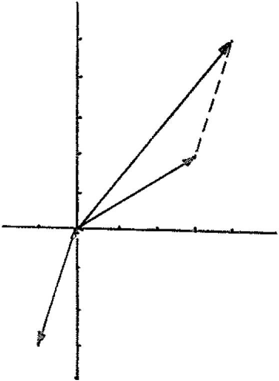
9i
8i
7i
6i
5i
4i
3i
2i
i
1 2 3 4 5 6 7 8 9 10 11

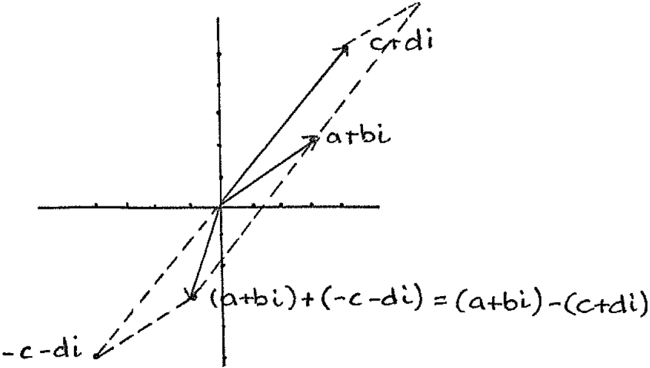

3+2i
Their sum:
7 - 3i
4-5i

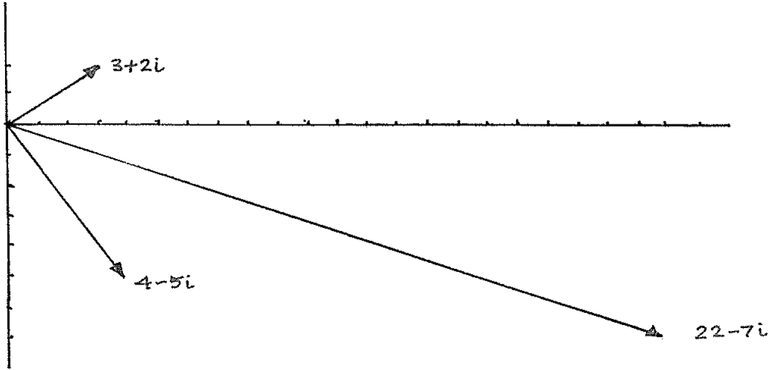




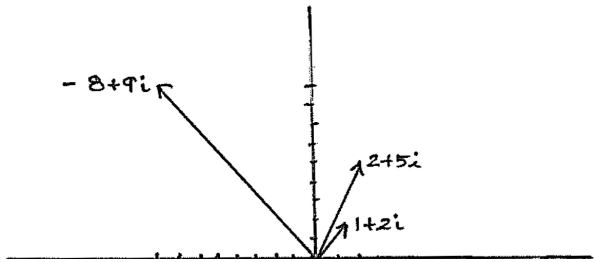

4+5i
3+2i
-1-3i



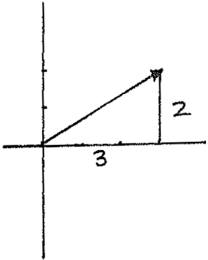

c+di
a+bi
(a+bi)+(-c-di) = (a+bi)-(c+di)
-c-di

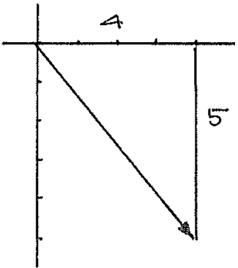


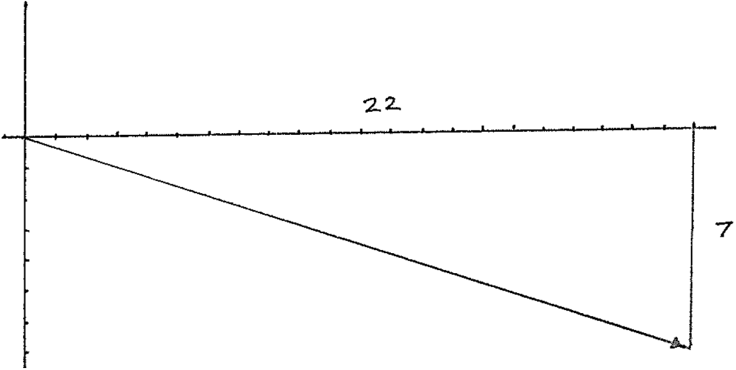
3+2i
4-5i
22-7i

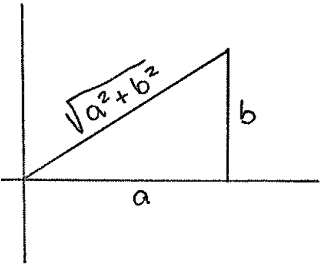


- 8+9i
2+5i
1+2i

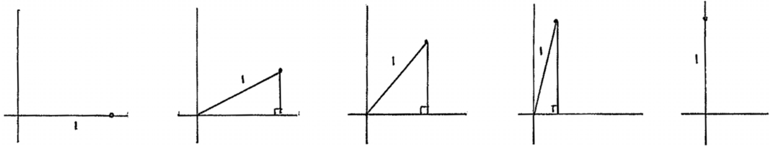

2
3

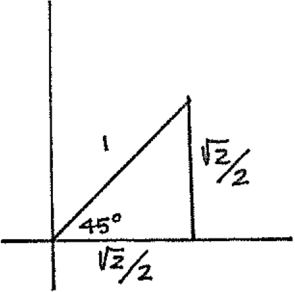

4
5

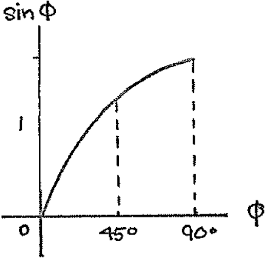

22
7


$\sqrt{a^2+b^2}$
b
a

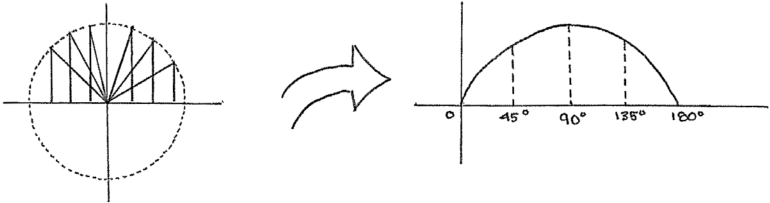
(a,b)
b
(0,0)
a
(a,b)
r
ϕ


1
1
1
1
1

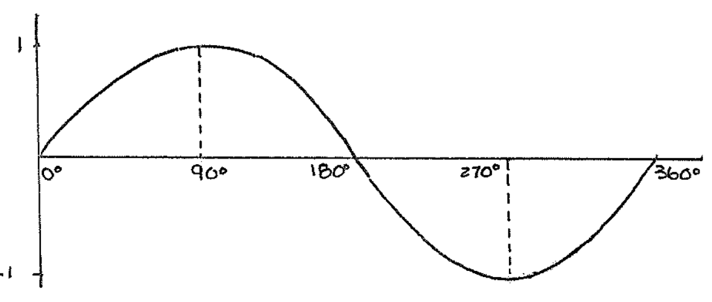

1
45°
√2/2
√2/2

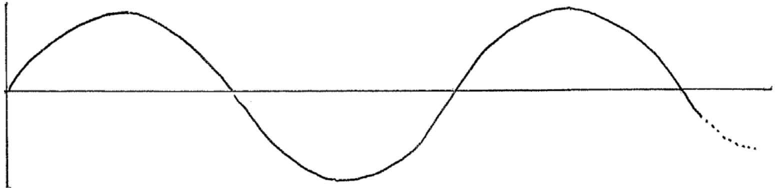


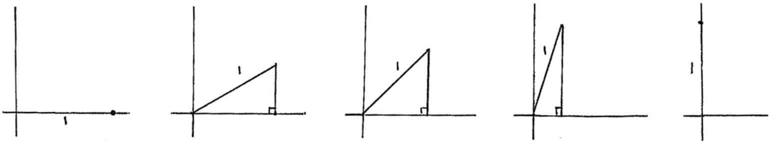
sin Φ
1
Φ
0
45°
90°

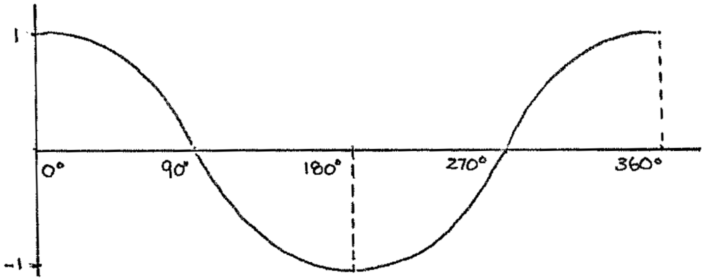


0
45°
90°
135°
180°

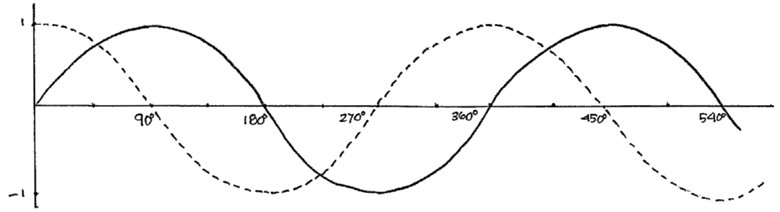

1
0°
90°
180°
270°
360°
-1



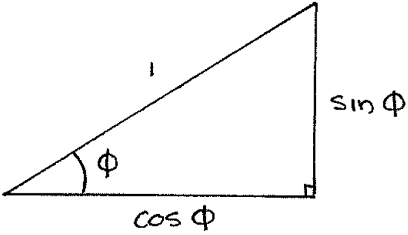

1
1
1
1
1



1
-1
0°
90°
180°
270°
360°



1
-1
90°
180°
270°
360°
450°
540°



$$\sin^2 \phi + \cos^2 \phi = 1.$$


1
b
ϕ
a
1
i sinϕ
ϕ
cosϕ

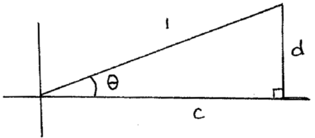
$$a = r \cos \phi \qquad\qquad b = r\, i \sin \phi.$$

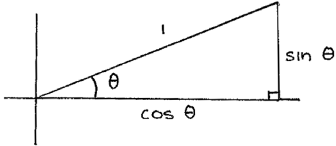
What was (a,b) is now $(r \cos \phi, r\, i \sin \phi)$, so

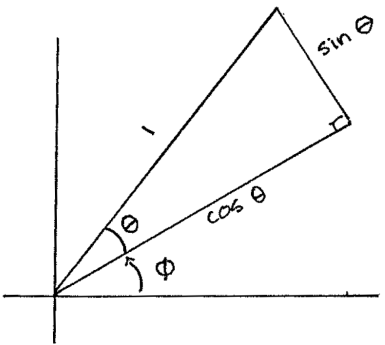
$$a + bi = r \cos \phi + r\, i \sin \phi$$

or more economically,

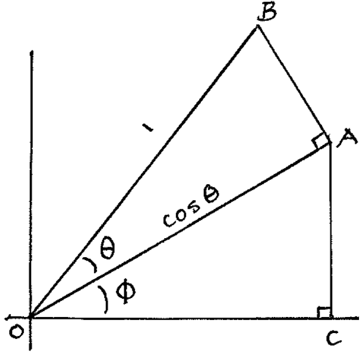
$$a + bi = r\,(\cos \phi + i \sin \phi).$$


1
d
θ
c

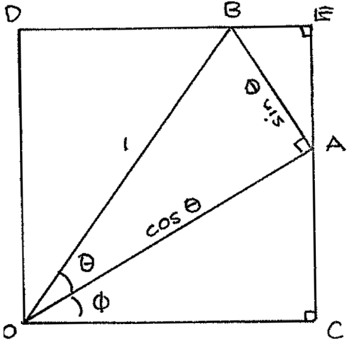

1
sin θ
θ
cos θ



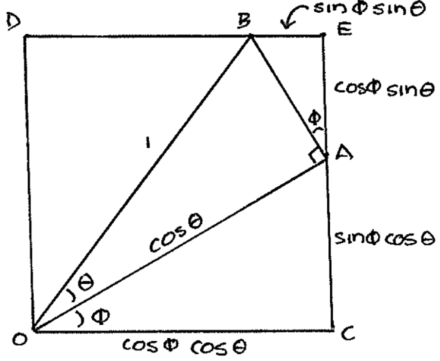
sin θ
1
θ
cos θ
φ

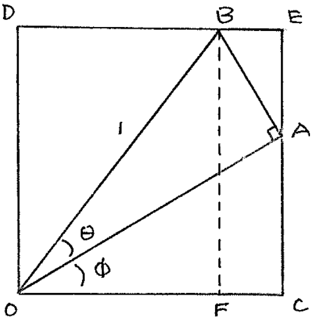


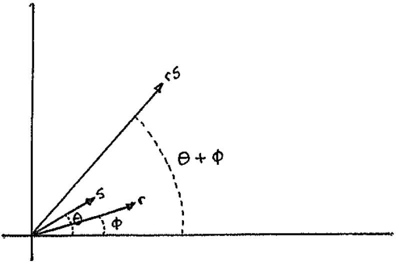
B
A
1
cos θ
θ
φ
O
C

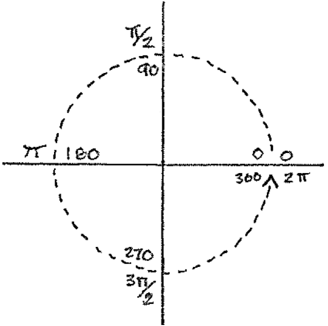

D
B
E
A
1
sin θ
cos θ
θ
φ
O
C

$$AE = \cos\phi \sin\theta.$$



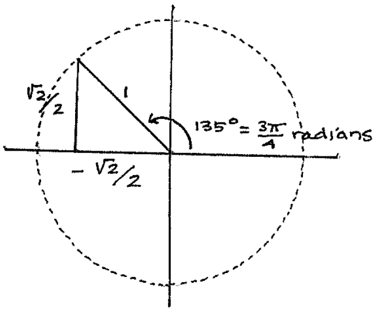

D
B
E
1
A
θ
ϕ
O
F
C


rs
θ + φ
s
r
θ
φ



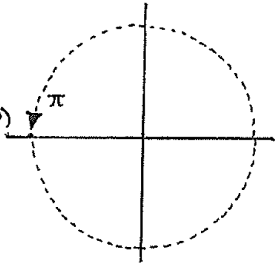
π/2
90
π
180
0
0
360
2π
270
3π/2

x in degrees	x in radians	sine x	cosine x
0	0	0	1
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
90	$\frac{\pi}{2}$	1	0
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$
180	π	0	–1
225	$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2}$
270	$\frac{3\pi}{2}$	–1	0
315	$\frac{7\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
360	2π	0	1
405	$\frac{9\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

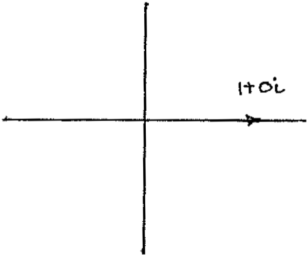


$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

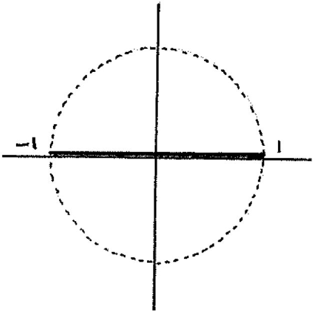
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

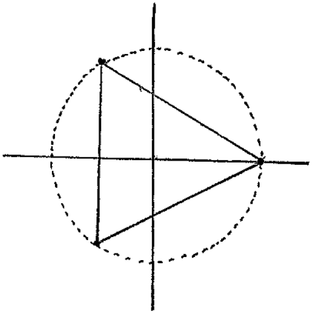


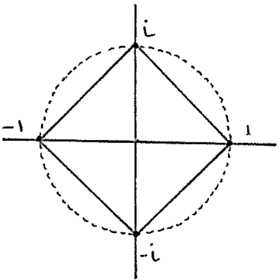
(-1,0)
π

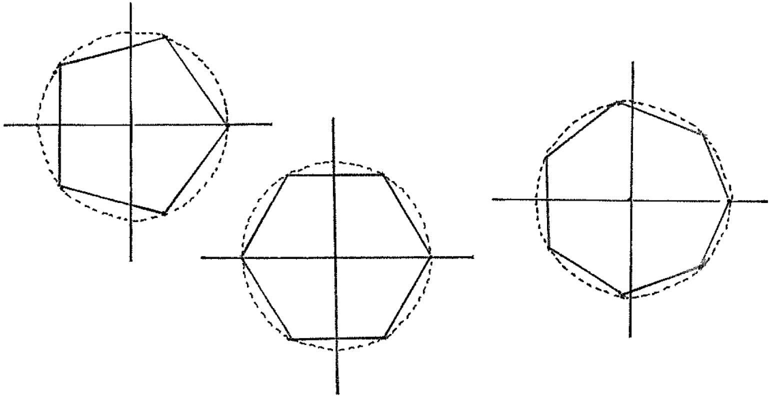


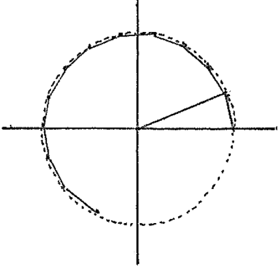
$r = 1$ and $\phi = 0$


-1
1

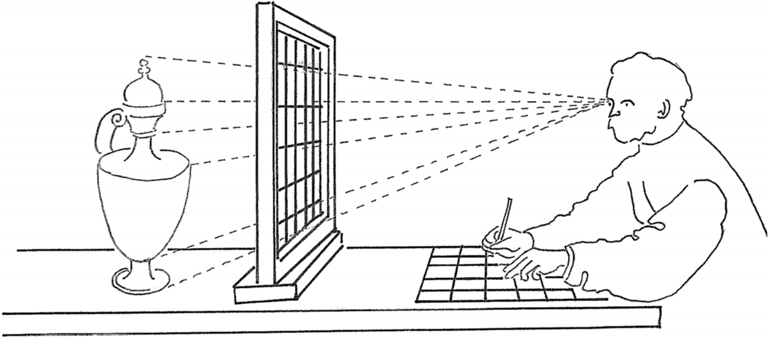


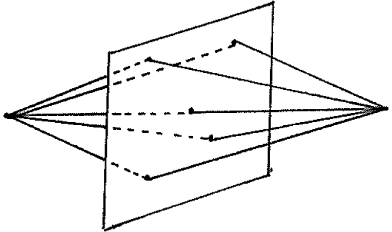

i
-1
1
-i

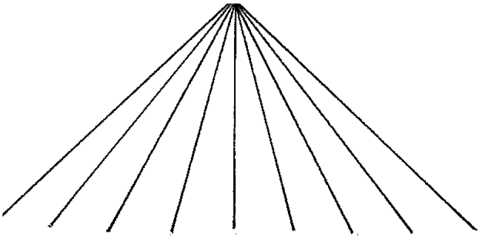


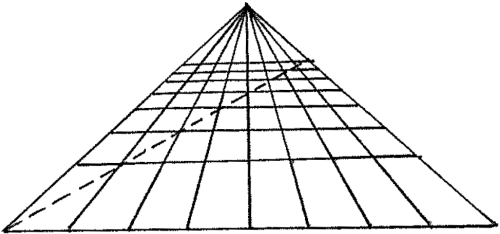


The n nth roots of unity, making angles of $\frac{2k\pi}{n}$ *from the horizontal root at 1, for each k from 0 to n – 1.*

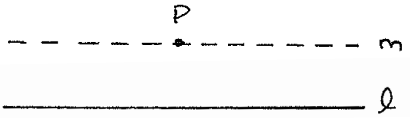


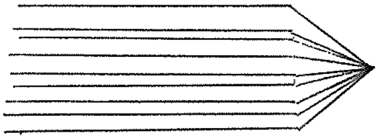


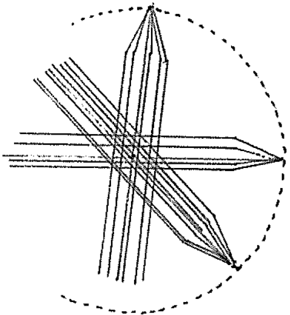


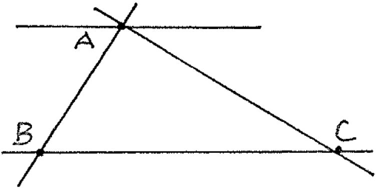


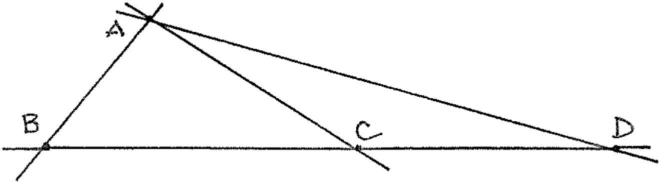
Poncelet (1788–1867). Loyal to his youth, he published in age his early work unedited by hindsight; loyal to France, he wasted his geometric foresight on its bureaucracy.

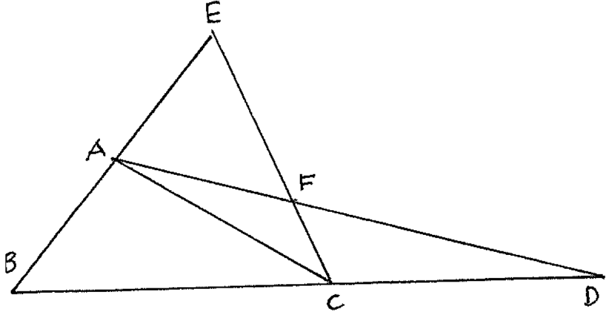

P
m
ℓ

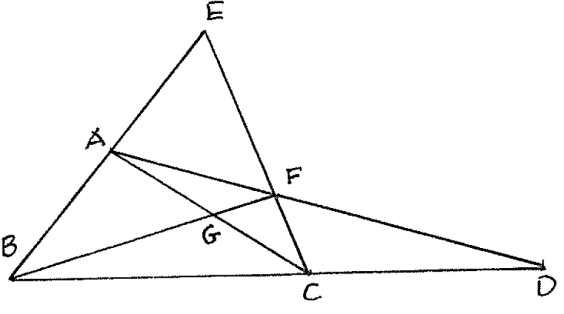


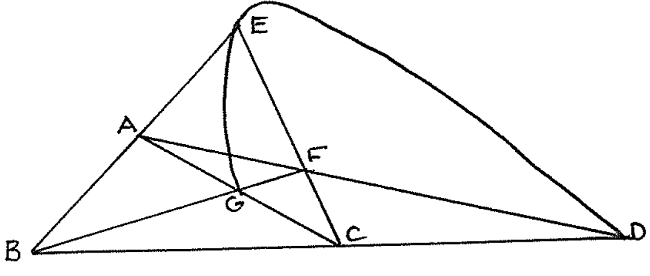


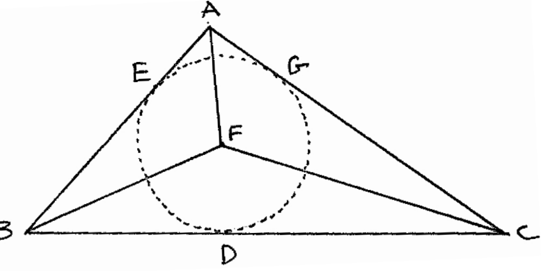

A
B
C


A
B
C
D


E
A
F
B
C
D


E
A
F
G
B
C
D


E
A
F
G
B
C
D


A
E
G
F
B
C
D

Line 1 = {a b c d}

Line 2 = {a e f g}

Line 3 = { b e h i}

Line 4 = { b f j k}

Line 5 = { b g l m}

Line 6 = { c e j l}

Line 7 = { c f h m}

Line 8 = { c g i k}

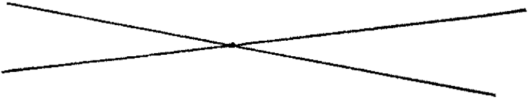
Line 9 = {d e k m}

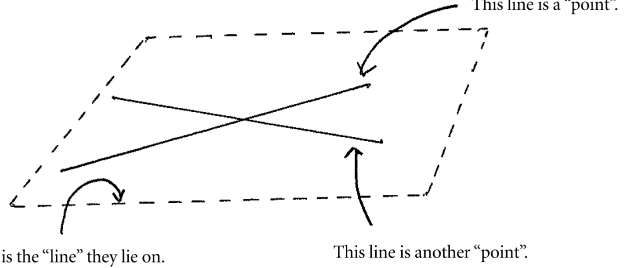
Line 10 = {d f i l}

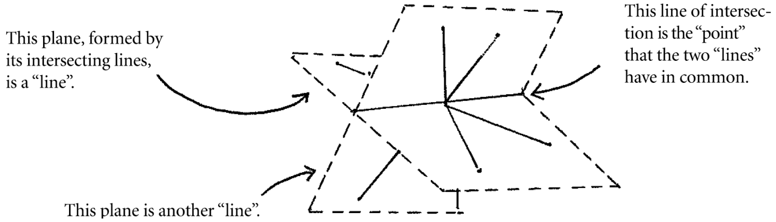
Line 11 = {d g h j}

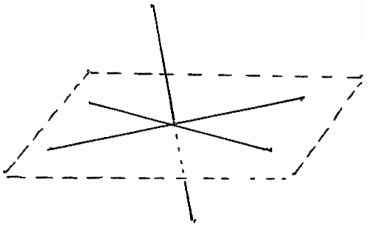
Line 12 = {a h k l}

Line 13 = {a i j m}

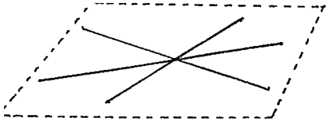


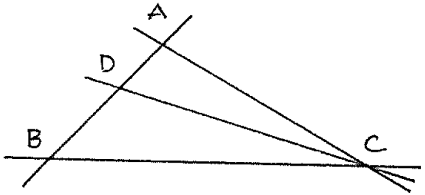

This line is a "point".
This plane is the "line" they lie on.
This line is another "point".

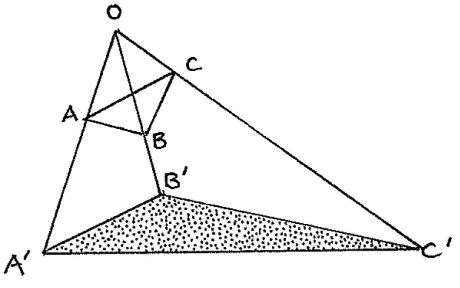

This plane, formed by its intersecting lines, is a "line".
This line of intersection is the "point" that the two "lines" have in common.
This plane is another "line".

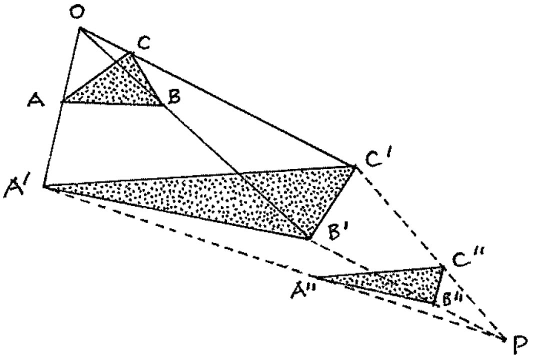


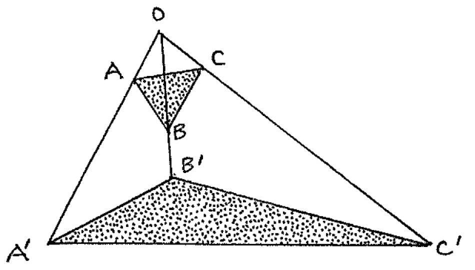
Three concurrent lines not all in the same plane—i.e., three “points” not all on the same “line.”



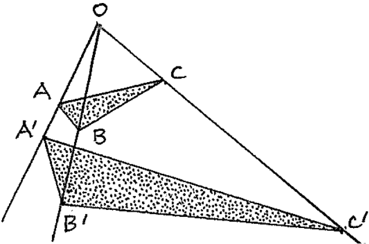

A
D
B
C

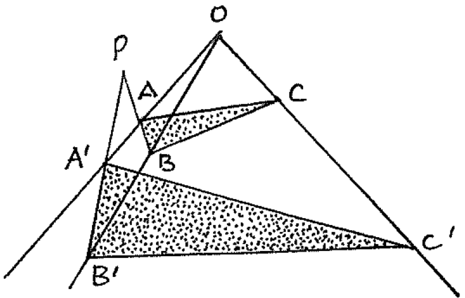

O
C
A
B
B'
A'
C'


O
C
A
B
C'
A'
B'
C''
A''
B''
P

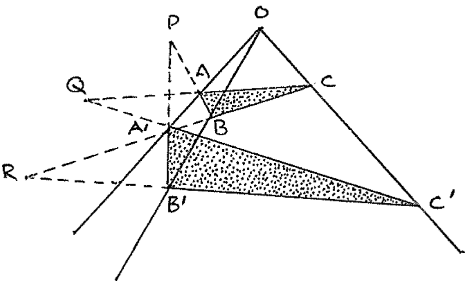


O
A
C
B
B'
A'
C'

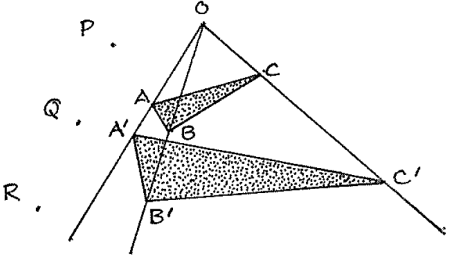

O
C
A
A'
B
B'
C'

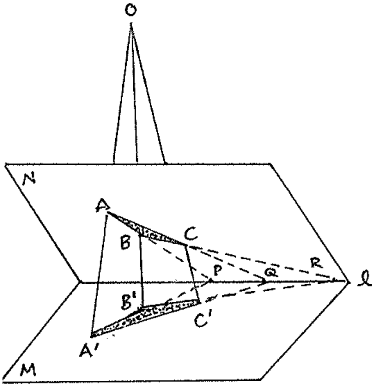


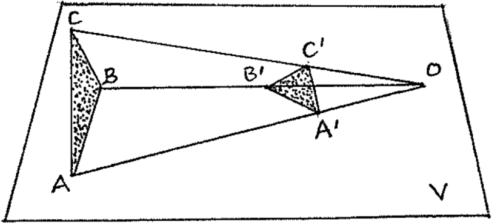
O
P
A
C
A′
B
C′
B′

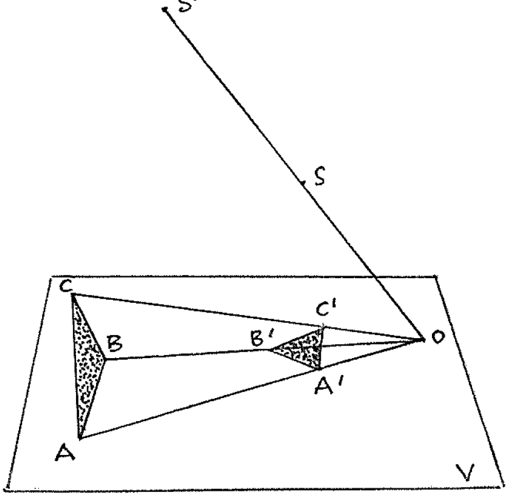


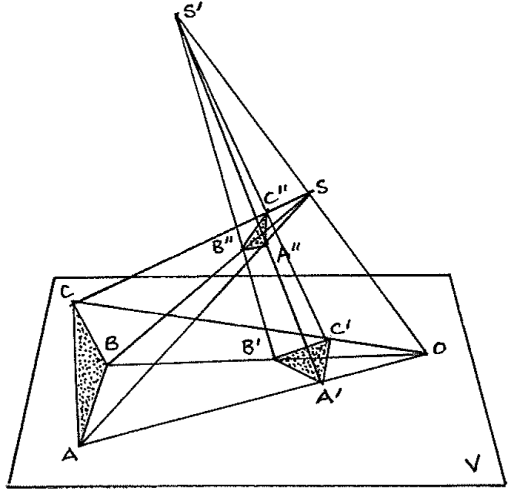
O
P
A
C
Q
A'
B
R
B'
C'

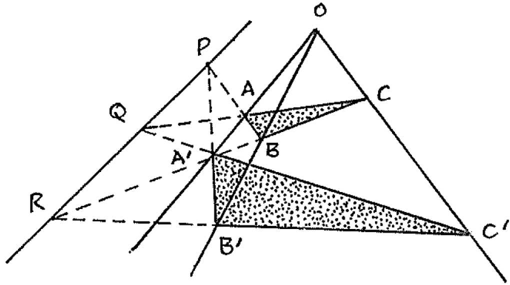

O
P
C
A
Q
A'
B
C'
R
B'

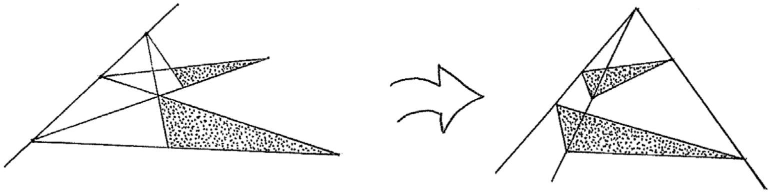

O
N
A
B
C
P
Q
R
l
B'
C'
A'
M

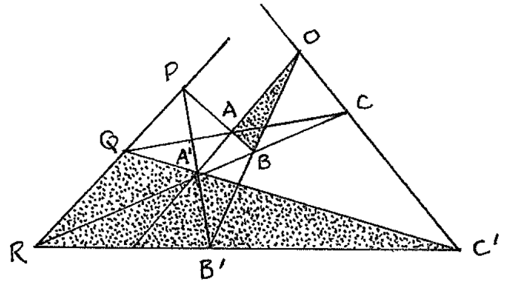

C
B
B'
C'
O
A'
A
V

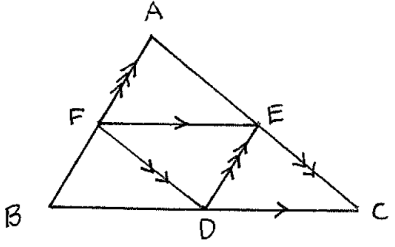

S'
S
C
C'
B
B'
O
A'
A
V

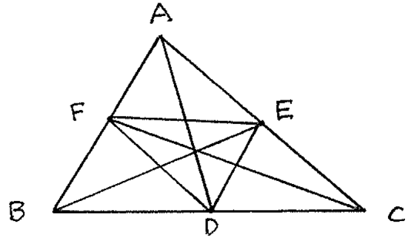

S'
S
C''
B''
A''
C
B
B'
C'
O
A'
A
V


O
P
A
C
Q
B
A'
R
B'
C'

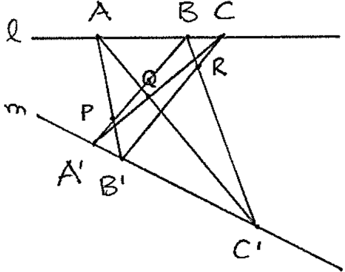



O
P
A
C
Q
A'
B
R
B'
C'

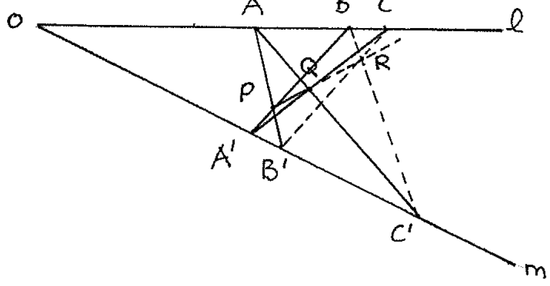

A
F
E
B
D
C

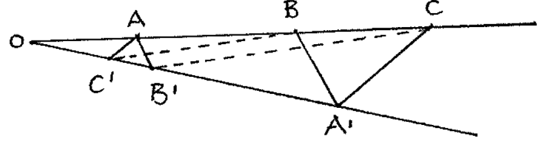


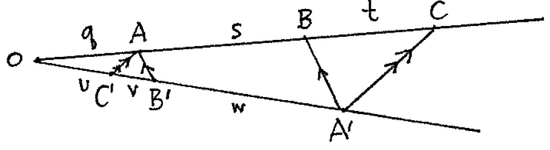
A
F
E
B
D
C



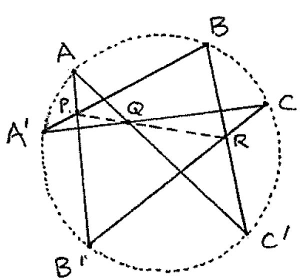
A
B
C
l
Q
R
m
P
A'
B'
C'

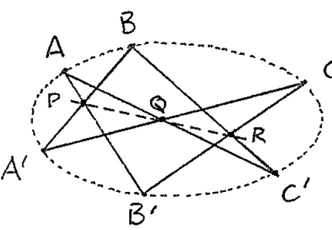

O
A
B
C
l
Q
R
P
A'
B'
C'
m


O
A
B
C
C'
B'
A'

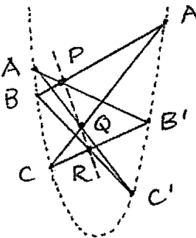


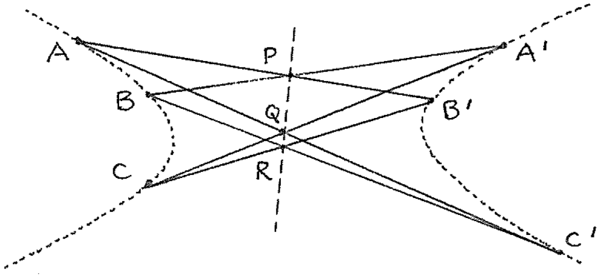
O
q
A
s
B
t
C
v
C'
v
B'
w
A'

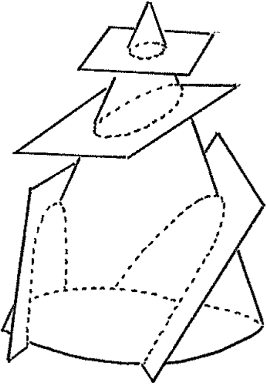

A
B
C
A'
B'
C'
P
Q
R

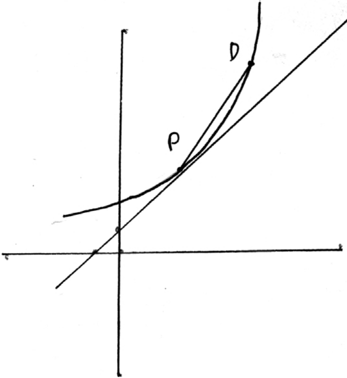


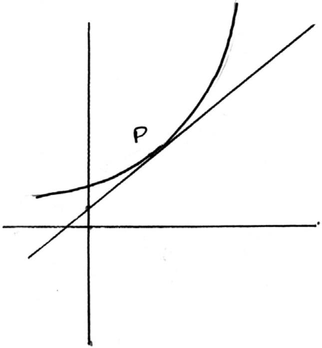
A
B
C
P
Q
R
A′
B′
C′


A'
P
A
B
Q
B'
C
R
C'


A
P
A'
B
B'
Q
C
R
C'




D
P


P

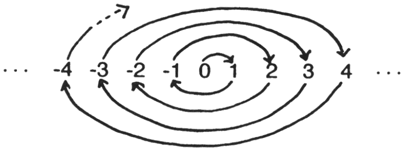
Cantor as a young man.

$$
\begin{array}{ccc}
\mathbb{N} & & 2\mathbb{N} \\
1 & \leftrightarrow & 2 \\
2 & \leftrightarrow & 4 \\
3 & \leftrightarrow & 6 \\
4 & \leftrightarrow & 8 \\
. & . & . \\
. & . & . \\
n & \leftrightarrow & 2n
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{N} & & 3\mathbb{N} \\
1 & \leftrightarrow & 3 \\
2 & \leftrightarrow & 6 \\
3 & \leftrightarrow & 9 \\
4 & \leftrightarrow & 12 \\
. & . & . \\
. & . & . \\
n & \leftrightarrow & 3n
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{N} & & 7\mathbb{N} \\
1 & \leftrightarrow & 7 \\
2 & \leftrightarrow & 14 \\
3 & \leftrightarrow & 21 \\
4 & \leftrightarrow & 28 \\
. & . & . \\
. & . & . \\
n & \leftrightarrow & 7n
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{N} & & m\mathbb{N} \\
1 & \leftrightarrow & m \\
2 & \leftrightarrow & 2m \\
3 & \leftrightarrow & 3m \\
4 & \leftrightarrow & 4m \\
. & . & . \\
. & . & . \\
n & \leftrightarrow & nm
\end{array}
$$


. . . -4 -3 -2 -1 0 1 2 3 4 . . .

$\mathbb{N}$		$\mathbb{Z}$
1	$\leftrightarrow$	0
2	$\leftrightarrow$	1
3	$\leftrightarrow$	−1
4	$\leftrightarrow$	2
5	$\leftrightarrow$	−2
6	$\leftrightarrow$	3
7	$\leftrightarrow$	−3
.	.	.
.	.	.

$$
\begin{array}{ccccccccccccc}
\frac{1}{1} & \rightarrow & \frac{2}{1} & & \frac{3}{1} & \rightarrow & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \frac{7}{1} & \frac{8}{1} & \frac{9}{1} & \cdots \\
& \swarrow & & \nearrow & & \swarrow & & & & & & & \\
\frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \frac{7}{2} & \cdots & & \\
\downarrow & \nearrow & & \swarrow & & & & & & & & & \\
\frac{1}{3} & & \frac{2}{3} & & \frac{3}{3} & & \frac{4}{3} & \frac{5}{3} & \cdots & & & &
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{N} & & \mathbb{Q}^{+} \\
1 & \leftrightarrow & \frac{1}{1} \\
2 & \leftrightarrow & \frac{2}{1} \\
3 & \leftrightarrow & \frac{1}{2} \\
4 & \leftrightarrow & \frac{1}{3} \\
5 & \leftrightarrow & \frac{3}{1} \\
6 & \leftrightarrow & \frac{4}{1} \\
7 & \leftrightarrow & \frac{3}{2} \\
8 & \leftrightarrow & \frac{2}{3} \\
9 & \leftrightarrow & \frac{1}{4} \\
10 & \leftrightarrow & \frac{1}{5} \\
11 & \leftrightarrow & \frac{5}{1} \\
 & \cdot &
\end{array}
$$

$$
\begin{array}{ccccccccccccccccc}
\frac{-5}{1} & \frac{-4}{1} & \leftarrow & \frac{-3}{1} & & \frac{-2}{1} & \leftarrow & \frac{-1}{1} & & \frac{1}{1} & \rightarrow & \frac{2}{1} & & \frac{3}{1} & \rightarrow & \frac{4}{1} & \frac{5}{1} \\
 & & \searrow & & \nwarrow & & \searrow & & & & \swarrow & & \nearrow & & \swarrow & & \\
\frac{-5}{2} & \frac{-4}{2} & & \frac{-3}{2} & & \frac{-2}{2} & & \frac{-1}{2} & & \frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \frac{4}{2} & \frac{5}{2} \\
 & & & & \searrow & & \nwarrow & \downarrow & & \downarrow & \nearrow & & \swarrow & & & & \\
 & & & \frac{-3}{3} & & \frac{-2}{3} & & \frac{-1}{3} & & \frac{1}{3} & & \frac{2}{3} & & \frac{3}{3} & & & \\
 & & & & & & \searrow & & & & \swarrow & & & & & & \\
 & & & & & \frac{-2}{4} & & \frac{-1}{4} & & \frac{1}{4} & & \frac{2}{4} & & & & & \\
 & & & & & & & \frac{-1}{5} & & \frac{1}{5} & & & & & & &
\end{array}
$$

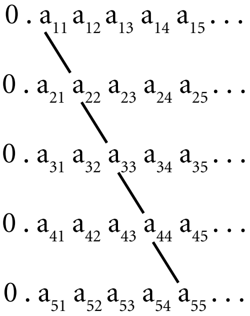
$$
\begin{array}{ccc}
\mathbb{N} & & \mathbb{Q} \\
1 & \leftrightarrow & 0 \\
2 & \leftrightarrow & \frac{1}{1} \\
3 & \leftrightarrow & \frac{-1}{1} \\
4 & \leftrightarrow & \frac{2}{1} \\
5 & \leftrightarrow & \frac{-2}{1} \\
6 & \leftrightarrow & \frac{1}{2} \\
7 & \leftrightarrow & \frac{-1}{2} \\
8 & \leftrightarrow & \frac{1}{3} \\
9 & \leftrightarrow & \frac{-1}{3} \\
10 & \leftrightarrow & \frac{3}{1} \\
11 & \leftrightarrow & \frac{-3}{1} \\
 & \cdot & \\
 & \cdot &
\end{array}
$$

$$1 \leftrightarrow 0\,.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\,\ldots$$

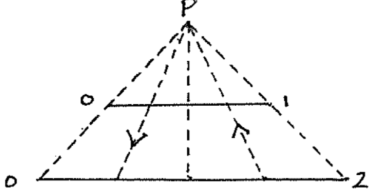
$$2 \leftrightarrow 0\,.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\,\ldots$$

$$3 \leftrightarrow 0\,.\,a_{31}\,a_{32}\,a_{33}\,a_{34}\,\ldots$$

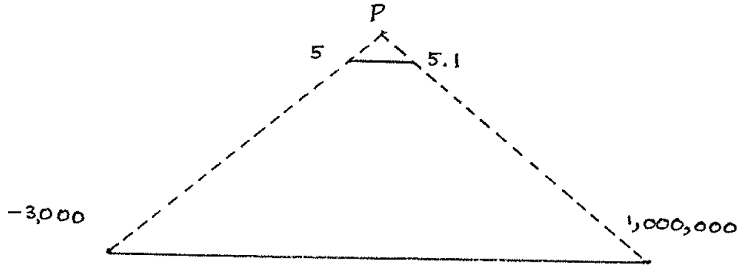
$$4 \leftrightarrow 0\,.\,a_{41}\,a_{42}\,a_{43}\,a_{44}\,\ldots$$

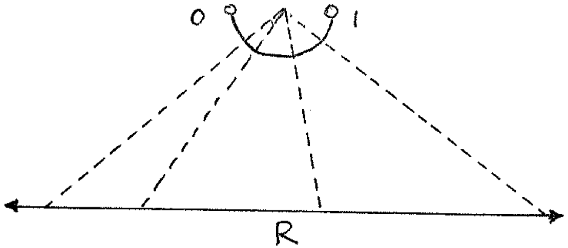

0 . a11 a12 a13 a14 a15 . . .
0 . a21 a22 a23 a24 a25 . . .
0 . a31 a32 a33 a34 a35 . . .
0 . a41 a42 a43 a44 a45 . . .
0 . a51 a52 a53 a54 a55 . . .

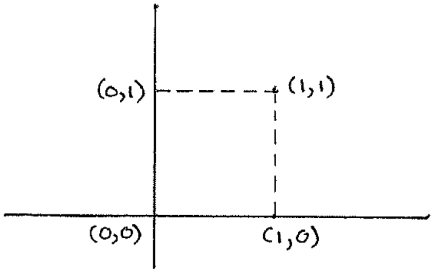
Cantor in middle age.

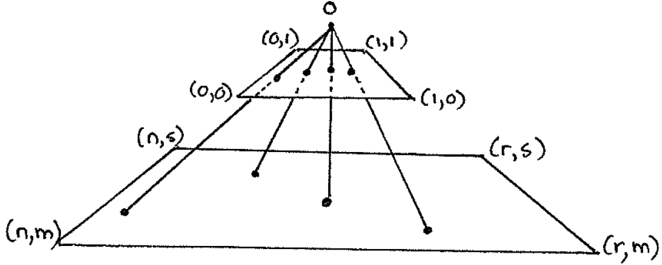


P
0
1
0
2

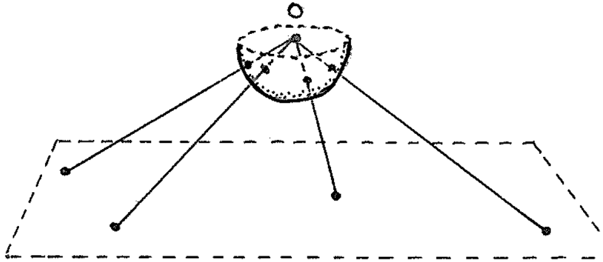

P
5
5.1
-3,000
1,000,000


0
1
R


(0,1)
(1,1)
(0,0)
(1,0)



O
(0,1)
(1,1)
(0,0)
(1,0)
(n,s)
(r,s)
(n,m)
(r,m)



1

1 1

1 2 1

1 3 3 1

•

•

Elements of S		Elements of $\mathscr{P}$S: The Set of All Subsets of S
f	↔	{g, j}
g	↔	Ø
h	↔	S
i	↔	{every element in S except i}
j	↔	{g, h, j, l}
k	↔	{k}
l	↔	{g}
.	.	.
.	.	.

Elements of S		Elements of $\mathscr{P}$S: The Set of All Subsets of S
.	.	.
.	.	.
w	$\leftrightarrow$	M
.	.	.
.	.	.

$1, 2, 3, \ldots$ $\qquad$ ω

$\omega + 1, \omega + 2, \omega + 3, \ldots$ $\qquad$ $\omega + \omega$

$\omega + \omega + 1, \omega + \omega + 2, \omega + \omega + 3, \ldots$ $\qquad$ $\omega + \omega + \omega$

$$\begin{array}{l} \omega \\ \omega + \omega \\ \omega + \omega + \omega \\ \vdots \\ \omega \cdot \omega = \omega^2 \\ \omega^3 \\ \vdots \\ \omega^\omega \\ \vdots \end{array}$$

<u>Ordinal</u>	a	1	2	3 …
	↕	↕	↕	↕
<u>Cardinal</u>	1	2	3	4 …

<u>Ordinal</u>	1	2	3	…	a_1	a_2	a_3	…
	↕	↕	↕		↕	↕	↕	
<u>Cardinal</u>	1	3	5	…	2	4	6	…

$\omega, \omega + 1, \omega + 2 \ldots$

•

•

•

4

3

2

Cantor, a few months before his death.