How Numbers Work

Discover the strange and beautiful world of mathematics

NEW SCIENTIST

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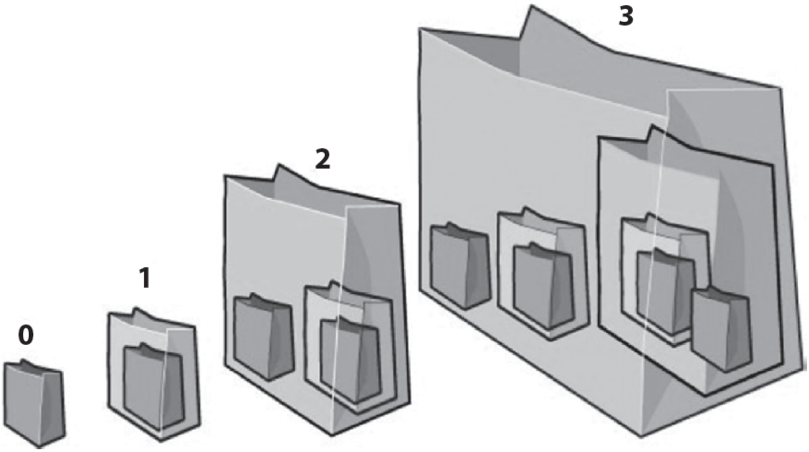
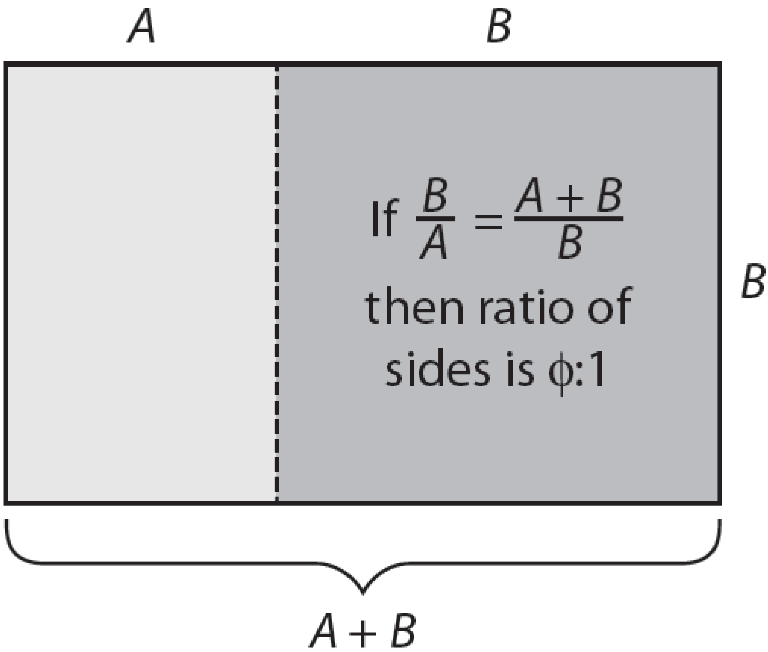
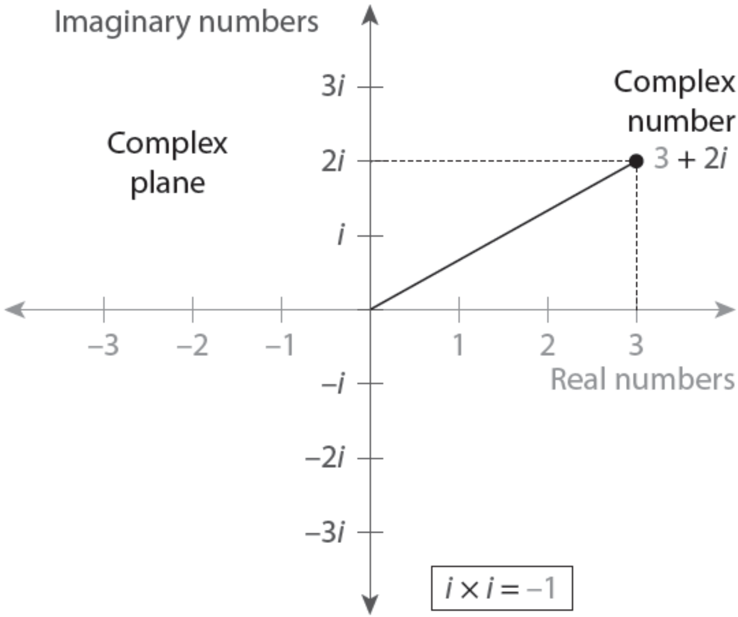


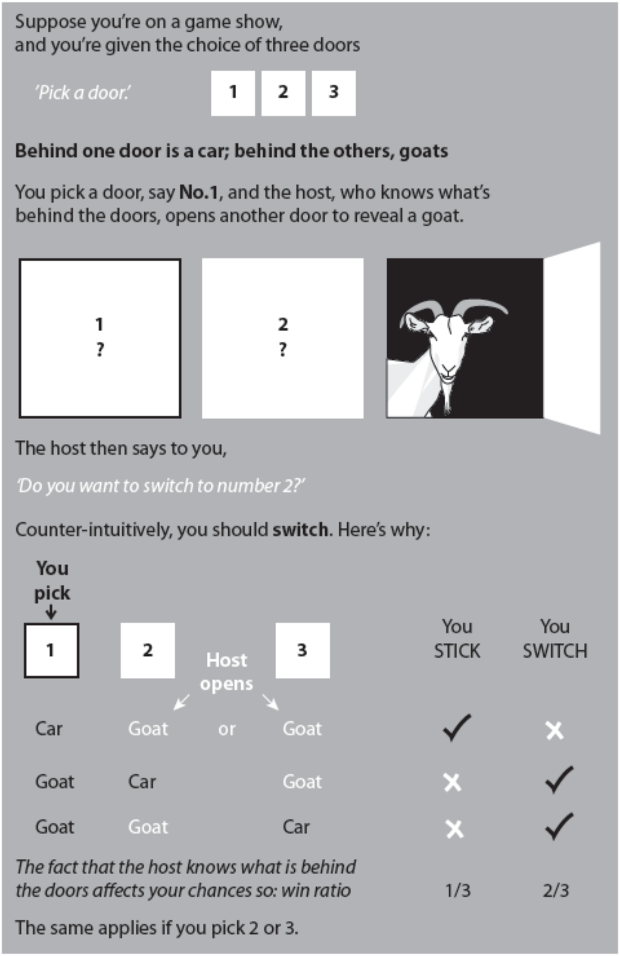
The empty set has no member, like an empty paper bag.
But by putting the empty paper bag in a larger paper bag you can form big and bigger sets – the basis of our definition of number.



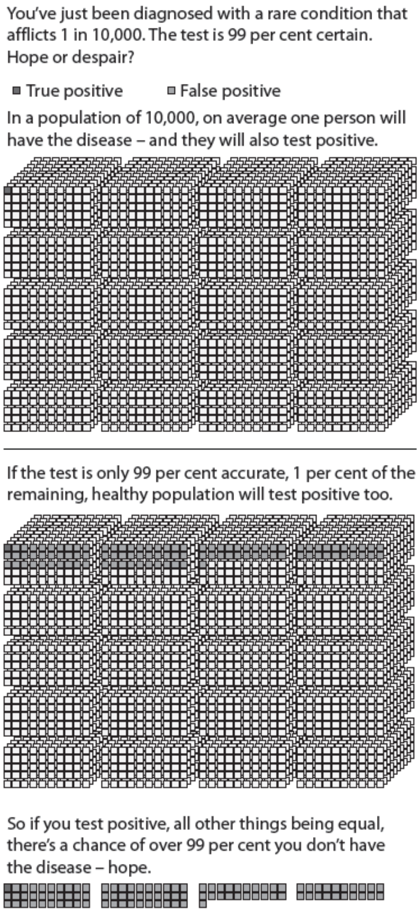
Shapes whose proportions conform to the golden ratio, ϕ, are supposedly particularly aesthetically pleasing.



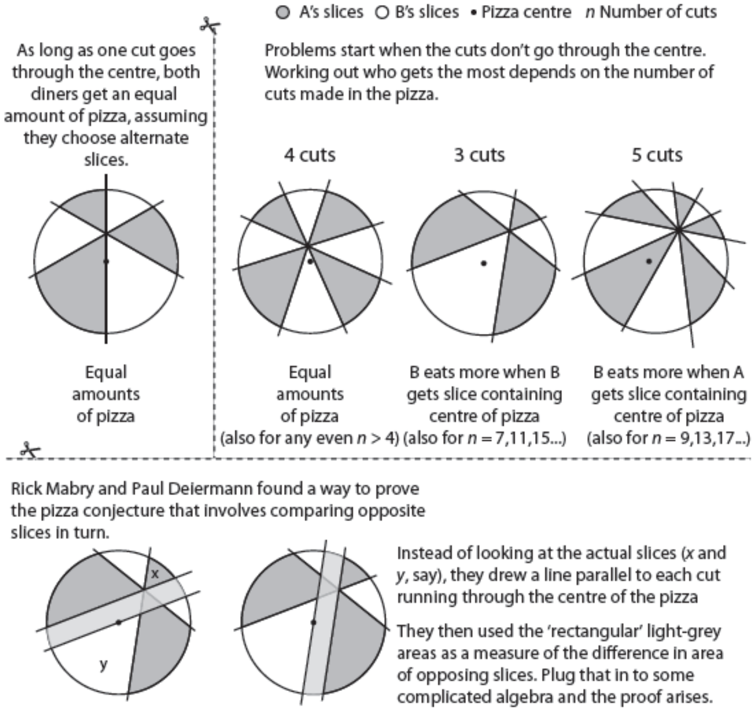
Complex numbers, which contain both real and imaginary elements based on the square root of -1, known as *i*, lie somewhere on a 2D ‘complex plane’.



The Monty Hall problem illustrates the counter-intuitive nature of some results in probability.

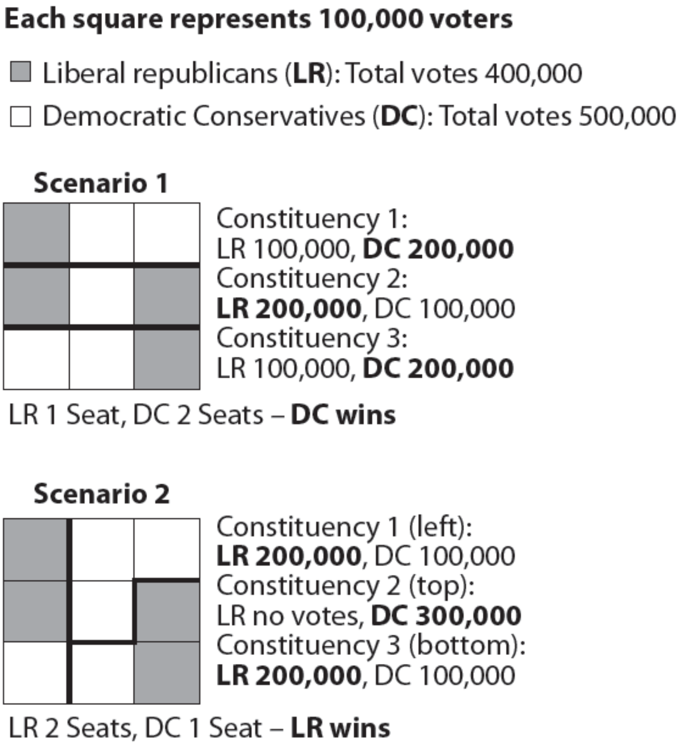


False positives in screening tests may lead to a false assessment of how likely you are to have a disease.

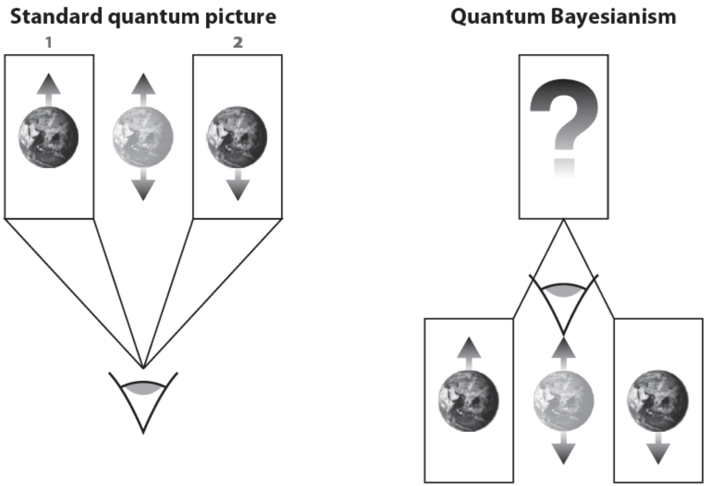


The 'pizza conjecture' asks who will get the biggest portion of pizza, assuming that diners A and N take alternate slices and that the angles between adjacent cuts are all equal.

The term ‘gerrymandering’ was inspired by a convoluted division of voting districts in nineteenth-century Massachusetts.



In a first-past-the-post voting system, where boundaries fall can determine the outcome.



Objects in the quantum world exist in a fuzzy combination of states. The act of measuring forces them to adopt a specific state (**1** or **2**).

The quantum states are all in our minds – they are just a fluid tool we use to understand our variable experiences of the world.

Quantum Bayesianism is an alternative to the standard picture of the quantum world. It uses Bayesian statistics to say its apparent uncertainty is all in our minds.

Forty-nine ideas

This section offers 7^2 extra ideas for how to explore the world of numbers in greater depth.

Seven places of mathematical pilgrimage

1 **The National Museum of Mathematics in New York**, located on East 26th Street between 5th and Madison Avenues, bills itself as the only museum dedicated to maths – or math – in North America. Its exhibits include a square-wheeled tricycle that rolls smoothly on a specially designed surface.

2 **The Winton Gallery at London's Science Museum**, which opened in 2016, is dedicated to maths. Its swirly ceiling, designed by the star architect Zaha Hadid, represents the mathematical equations that describe air flow.

3 **The Seven Bridges of Königsberg**. Only one of these still stands – most of the city, now the Russian enclave of Kaliningrad sandwiched between Lithuania and Poland on the Baltic Sea, was destroyed during the Second World War. Leonhard Euler's proof that you could not cross all seven just once without backtracking is widely considered to be the founding of the discipline of graph theory. The word is that the present bridge configuration makes it possible: perhaps it is worth a walking tour on the Baltic to find out.

4 **The old city of Syracuse in Sicily, Italy**, is a World Heritage Site and well worth a visit. For maths buffs, it was the scene of Archimedes' apocryphal 'Eureka!' moment when he realized the mathematical law of displacement in his bathtub. Equally unverified is the story that Archimedes met his end when a Roman soldier, part of an army that had just sacked Syracuse, came across him drawing diagrams in the sand – and, against the orders of his general, stabbed him.

5 **Broom Bridge, on the Royal Canal in Dublin's northern suburbs**, has a plaque with the equation for four-dimensional complex numbers, or quaternions – celebrating the location where mathematician William Rowan Hamilton was inspired to create them.

6 **Tilings at the Alhambra in Granada, Spain**, show complex periodic patterns – in fact, they contain examples of 13 of the 17 classes of periodic symmetry. For fans of less regularity, the terrace outside the Andrew Wiles Building of the Mathematical Institute in Oxford, UK, is paved with an aperiodic Penrose tiling.

7 **Göttingen, Germany**, was the most significant place in the development of modern mathematics. There is nothing of particular mathematical interest to see there today, but in the nineteenth and early twentieth century its university was home to a series of seminal mathematicians, among them Carl Friedrich Gauss, David Hilbert, Emmy Noether and Bernhard Riemann.

Seven quirky integers

1 **1** It may be, unlike 0, indisputably a number, but 1 has properties that make it stand out. Just as zero is the 'additive identity' – adding 0 to anything changes nothing – 1 is the 'multiplicative identity'. Any number multiplied by 1 doesn't change, including 1 itself. It follows that 1 is the only number that is its own square, its own cube and so on. It is also the only natural number that is neither a prime number (it is divisible only by 1, so falls down on that definition) nor a 'composite' number that can be produced by multiplying two smaller natural numbers together.

2 **6** In his mathematical primer *The Elements*, Euclid coined the term 'perfect number'. It refers to a number that is the sum of all the numbers it divides by, excluding itself. Thus 6 = 1 + 2 + 3 is the first example; the next are 28, 496 and 8128.

3 **70** The quirkiness of this number speaks for itself: it is the smallest 'weird number'. A weird number has two qualities. First, it is an 'abundant' or 'excessive' number, meaning that the sum of all the numbers it divides by, including 1 but not including itself, is bigger than itself: in the case of 70, 1 + 2 + 5 + 7 + 10 + 14 + 35 = 74. But a weird number is also not 'semi-perfect'; that is to say, no subset of those divisors adds up to the number itself. This is a rare combination – after 70, the next examples are 836 and 4030.

4 **1729** This number would have been seen as unremarkable were it not for an anecdote told by the mathematician G. H. Hardy about his friend and mentee Srinivasa Ramanujan, to illustrate the latter's peculiar brilliance.

Hardy wrote: 'I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.' Those ways are $1^3 + 12^3$ and $9^3 + 10^3$, and 1729 has since been known as the Hardy-Ramanujan number.

5 **3435** This is a Münchhausen number, one of only two known to exist. Named after the German nobleman Baron von Münchhausen who was known for his tall tales, 3435 can 'raise itself' – it is the sum of its digits raised to their own power. $3^3 + 4^4 + 3^3 + 5^5 = 27 + 256 + 27 + 3125 = 3435$. The other Münchhausen number is, of course, 1.

6 **6174** Take any four-digit number that contains at least two different digits. Arrange the digits in descending and ascending order, and subtract the second from the first. Repeat the previous step with this new number (treating any zeros as normal digits), until the result of the subtraction is 6174. As the Indian mathematician D. L. Kaprekar noted in 1955, this will happen in no more than seven steps.

7 **Graham's number** In the 1970s mathematician Ronald Graham was working on a problem to do with cubes in higher dimensions. When he finally got there, the answer involved a number that was not infinite but so large that it could not be written down – there is literally not enough space in the universe. It follows that we cannot reproduce it here, although we do know that its last digit is 7.

Seven (seeming) paradoxes

1 **The coastline paradox** How long is the coast of Britain? That was the question mathematician Benoit Mandelbrot asked in 1969, in a paper with the subtitle 'Statistical self-similarity and fractional dimension'. In it, he explored an apparent paradox. It is that the length of a coastline of an island such as Great Britain, with its many inlets, curves and complications on all sorts of scales, depends on the length of the thing you use to measure it. The smaller the ruler, the longer it becomes as you take into account more and more ins and outs. Yet it clearly does have a set length, doesn't it?

For Mandelbrot, the answer was that it makes no sense to treat a shape like a coastline with repeating complications on many scales as a straight line with a length in just one dimension. On the other hand, a coastline is clearly not a shape in two dimensions, either. It is somewhere in between – it has the counter-intuitive property of having a fractional dimension. In the word Mandelbrot introduced to describe such patterns in 1975, he kick-started a whole new avenue of mathematical discovery, 'fractals'.

2 **Zeno's paradox** Motion is impossible and change does not happen. That is the lesson of a series of paradoxes devised by the Greek philosopher Zeno of Elea in the fifth century BCE. The best known is that of Achilles and the tortoise. Achilles gives a tortoise a head start, say of 100 metres, in a race. After a certain time, Achilles has run 100 metres, in which time the tortoise has moved 10 metres, so is still ahead. But in the time it

takes Achilles to move that extra 10 metres, the tortoise has moved on again. In fact, you can show that, although Achilles will get very close to the tortoise, he will never actually overtake it.

Only in the late nineteenth century, with the application of some nifty calculus and a full appreciation of the mathematics of the infinite series that the problem represents, did we have what appears to be a mathematically watertight resolution of the paradox that conforms with experience: yes, Achilles can overtake the tortoise.

3 **The paradox of the heap** Also known as the Sorites paradox, this highlights the importance to any mathematical argument of precise, logically defined terms. You have a heap of grains of sand, which you remove one by one. Removing any one grain of sand does not turn the heap into a non-heap. But then again, a single grain of sand is not a heap. So if you keep on removing grains, when does it turn from a heap to a non-heap? It sounds trivial, but resolutions tend to require either setting an arbitrary numerical boundary of the size of a heap, denying that heaps can exist in the first place, or introducing a new three-valued logic that allows states of heap, non-heap and neither one nor the other.

4 **The elevator paradox** The cosmologist George Gamow had an office on the second floor of his building, while his colleague Marvin Stern was on the sixth floor, near the top. The two observed an odd fact: whereas the first elevator that came to Gamow's floor was almost always going down, the first to arrive at Stern's was almost invariably going up – even though (unless a continual

flow of elevators was originating somewhere in the middle of the building) just as many elevators should have been going up as down at each floor. This is, in fact, a real effect – but only some complex modelling of where an elevator spends most of its time in a building could explain why it should be the case.

5 **The friendship paradox** Most people have fewer friends than their friends have, on average. What appears to be a paradox is also a real effect, and has to do with the structure of the sort of networks that permeate our social circles. In simple terms, people with a large number of friends have a greater likelihood of being in your own group of friends, skewing the average. A similar effect means that, on average, most people's partners have had a greater number of other sexual partners.

6 **Simpson's paradox** In 1973 an analysis of admissions to the graduate school at the University of California, Berkeley, showed that men applying were more likely to be admitted than women. But when the people conducting the analysis broke it down by individual departments, there were more departments with a significant bias towards women. It is a famous example of Simpson's paradox, in which a trend seen in different groups of data disappears when those groups are combined. It is the curse of many a medical trial when researchers are trying to work out whether the effect of a drug is real across a whole population, say. In the UC Berkeley case, it turned out that the phenomenon was accounted for by more women applying to competitive departments with low general rates of admission than to any pervasive gender bias.

7 **Gabriel's horn paradox** By taking a graph of a particular mathematical function ($f(x) = 1/x$ for the domain $x > 1$) and rotating it in three dimensions around the x-axis, it is possible to create a shape that, although it has an infinite surface area, has a finite volume. Known as Gabriel's horn, or Torricelli's trumpet, the resulting shape would be practically puzzling: it could, for example, hold only a finite amount of paint while requiring an infinite amount of paint to coat its surface.

Seven lesser-known great mathematicians

1 **Muhammad ibn Mūsā al-Khwārizmī** (*c.*780–*c.*850) was a Persian mathematician whose writings, once translated into Latin, transformed Western mathematics, introducing among other things the decimal number system. The word 'algebra' derives from *al-jabr*, an operation he used to solve quadratic equations. The Latinized form of his name, Algoritmi, gives us the word 'algorithm'.

2 **Gerolamo Cardano** (1501–76) was a polymathic mathematician who, among other things, was the first to make use of imaginary numbers. He was also a compulsive gambler – a trait that led him to write the first systematic study of probability.

3 **Carl Friedrich Gauss** (1777–1855) is one of history's most influential mathematicians, making wide-ranging contributions to fields from number theory to statistics. He was an obsessive perfectionist and many of his results were not published in his lifetime. Today he is perhaps best known for the Gaussian or normal distribution in statistics, which predicts how a random quantity – such as the height of all mathematicians – will group around an average value.

4 **Evariste Galois** (1811–32) founded several branches of abstract algebra, as well as laying the groundwork for group theory. He did all this while being a radical adherent of French revolutionary ideals, dying in mysterious circumstances in a duel at the age of just 20.

5 **Emmy Noether** (1882–1935) was described by Albert Einstein as 'the most significant creative mathematical genius thus far produced since the higher education of women began'. Others have argued that only the first part of the sentence is necessary. The theorem that bears her name – that mathematical symmetries translate into conserved physical quantities – has provided a roadmap for discoveries in fundamental physics. Denied a full professorship in Göttingen because she was a woman, as a Jew she died in exile in the USA, a victim of Nazi racial laws.

6 **John von Neumann** (1903–57) is often called the last of the universal mathematicians. He made seminal contributions to game theory and the development of computing, as well as contributing to the development of quantum theory and the nuclear bomb. He was also famed for his photographic memory, sometimes entertaining friends by reciting pages from the telephone directory and entire works of literature such as *A Tale of Two Cities*.

7 **Paul Erdős** (1913–96) was a Hungarian-born mathematician known as 'the oddball's oddball'. He led an itinerant life travelling from conference to conference, eschewing most possessions and showing up at colleagues' houses announcing, 'My brain is open.' His uniquely collaborative style led him to co-author more than 1500 papers in his lifetime, and to the concept of the Erdős number as a measure of a mathematician's standing in the field. If you have an Erdős number of zero, you are Erdős himself; if you have a number one, you have collaborated with him; if you have number two, you have collaborated with a collaborator of his … and so on.

Seven maths jokes

… with seven maths-type explanations.

1 Why did the chicken cross the Möbius strip?
To get to the same side.

(A Möbius strip is a topological form with only one side.)

2 Two statisticians go hunting. The first one fires at a bird but overshoots by a foot. The second one fires and undershoots by a foot. They high-five and say 'Got it!'

(It's the law of averages …)

3 Why do mathematicians like forests?
Because of all the natural logs.

(The natural log(arithm)s are those based on Euler's number, *e*.)

4 A: 'What is the integral of 1/cabin?'
B: 'Log cabin.'
A: 'No, houseboat – you forgot the C.'

(It's integral calculus – performing this operation on a function $1/x$ produces the (natural) logarithm of x as an answer – as long as you remember to add a 'constant of integration', C.)

5 A physicist, a biologist and a mathematician are sitting on a bench across from a house. They watch as two people go into the house, and then, a little later, three people walk out. The physicist says, 'The initial measurement was incorrect.' The biologist says, 'They must have reproduced.' And the mathematician says, 'If exactly one person enters that house, it will be empty.'

(Only a mathematician truly counts on negative numbers.)

6 Infinitely many mathematicians walk into a bar. The first says, 'I'll have a beer.' The second says, 'I'll have half a beer.' The third says, 'I'll have a quarter of a beer.' The barman pours just two beers. 'Is that all you're giving us?' the mathematicians ask. The bartender says: 'Come on, guys. Know your limits.'

(You can prove mathematically that the sum, or limit, of the infinite number of terms in the sequence $1/2^n$, so $1 + 1/2 + 1/4 \ldots$, is *2*.)

7 What is a polar bear?
A Cartesian bear after a coordinate transformation.

(Cartesian and polar coordinates are two alternative coordinate systems.)

Seven maths films

1 ***Good Will Hunting* (1997)** is a fictional story of Will Hunting, a janitor at MIT with genius-level mathematical ability – but who, to thrive, must first overcome his demons.

2 **π (1998)** is a horror-thriller whose protagonist is a number theorist who sees mathematical patterns in everything around him, with unpalatable results.

3 ***A Beautiful Mind* (2001)** is a dramatized account of the life of John Nash (1928–2015), an American mathematician and game theory pioneer who won both the Abel Prize and the Nobel Prize for Economics, despite suffering from paranoid schizophrenia.

4 ***Proof* (2005)** is a fictional story based on a Pulitzer prize-winning play. It centres on a dispute about the ownership of a proof discovered in a deceased mathematician's effects. Cambridge mathematician and Fields Medallist Timothy Gowers acted as a consultant.

5 ***Travelling Salesman* (2012)** is an intellectual thriller based around four mathematicians who discover a solution to the notorious 'P = NP?' problem of computational complexity (see Chapter 7) and the moral consequences of their discovery.

6 ***The Imitation Game* (2014)** is a historical drama film centred around Alan Turing and other mathematicians who decrypted the Nazi Enigma codes during the Second World War.

7 ***The Man Who Knew Infinity* (2015)** tells the true story of the Indian mathematical genius Srinivasa Ramanujan (1887–1920) and his remarkable collaboration with the British mathematician G. H. Hardy.

Seven ideas for further reading

1 **The Very Short Introductions series** published by Oxford University Press are small books written by experts for laypeople. It includes titles on mathematics, algebra, numbers, infinity, probability, statistics and logic, among others.

2 **'The unreasonable effectiveness of mathematics in the natural sciences'** is a seminal essay written in 1960 by the physicist Eugene Wigner. It is available at http://www.maths.ed.ac.uk/~aar/papers/wigner.pdf

3 *Mathematics Made Difficult* by Carl E. Linderholm (1971) is a book for those who appreciate the perverse side of mathematics. It consists of mathematical proofs for obvious statements – made as complex as possible.

4 **'100,000 digits of π'.** If you want to look for patterns in the digits of π, the website http://www.geom.uiuc.edu/~huberty/math5337/groupe/digits.html lists the first 100,000.

5 **'The largest known primes – a summary'** details the largest known prime numbers and the search for them. Go to http://primes.utm.edu/largest.html

6 **Wolfram MathWorld** is an extensive online maths resource giving definitions and background for a range of mathematical concepts. Go to http://mathworld.wolfram.com/

7 *New Scientist*'**s website** has an extensive archive of articles and is regularly updated on all themes mathematical and scientific. Go to www.newscientist.com